

GEOMETRIC SPACES FROM ARBITRARY CONVEX POLYTOPES

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Abstract

We associate a geometric space to an arbitrary convex polytope. Our construction parallels the construction by D. Cox of a toric variety as a GIT quotient [8]. The spaces that we obtain are endowed with a natural stratification and perfectly mimic the features of toric varieties associated to rational convex polytopes.

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Introduction

The goal of this paper is to construct geometric spaces from nonrational nonsimple polytopes that generalize toric varieties. To each n -dimensional convex polytope Δ there corresponds a unique fan, generated by the 1-dimensional cones dual to the facets of the polytope. The polytope Δ is *rational* if these 1-dimensional cones are generated by vectors lying in a lattice L . The polytope Δ , in a neighborhood of a p -dimensional face F , is the product of F by a cone over an $(n - p - 1)$ -dimensional polytope Δ_F . The face F is regular if Δ_F is a simplex, singular otherwise. The polytope Δ is *simple* if all of its vertices are regular, which implies that all of its faces are also regular. A simple nonrational convex polytope can always be perturbed into a rational one, combinatorially equivalent. This is not true in the nonsimple case: there are nonsimple convex polytopes that are not even combinatorially equivalent to rational ones (the first example, due to M. Perles, was published in the book by B. Grünbaum [11], see also [23] and [19]).

A rational convex polytope in a lattice L gives rise to a toric variety X , acted on by the torus $\mathfrak{d}_C/L \simeq (\mathbb{C}^*)^n$, where $\mathfrak{d} = L \otimes_{\mathbb{Z}} \mathbb{R}$. There is a one-to-one correspondence between p -dimensional orbits of the torus and p -dimensional faces of the polytope. The orbit corresponding to the interior of the polytope, $\mathfrak{d}_C/L \simeq (\mathbb{C}^*)^n$, is open and dense; its compactification X is obtained by gluing, to the n -dimensional orbit, the smaller orbits corresponding to the other faces of the polytope. The gluing pattern reflects the combinatorics of the polytope. This subdivision in orbits yields a natural

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stratification of X : the union of the orbits corresponding to the regular faces is the set of rationally smooth points of X , whilst the orbits corresponding to the singular faces are the singular strata.

We are interested in the following problem: is there an analogue of this construction for nonrational polytopes and what are the spaces we end up with?

When dealing with nonrational polytopes, the first step is to replace lattices with quasilattices, this was first done by E. Prato in [18], in order to generalize to nonrational simple convex polytopes the Delzant procedure [9]. A quasilattice Q in a vector space \mathfrak{d} is a \mathbb{Z} -submodule of \mathfrak{d} generated by a finite set of spanning vectors of \mathfrak{d} , it is an important notion in the physics of quasicrystals ([21],[22]). We will also need the notions of *quasitorus* and *quasifold*, introduced in [18]. Quasifold is a generalization of manifold and orbifold: the local models are quotients of manifolds modulo the smooth action of discrete groups, not necessarily finite. Therefore a quasifold might be non Hausdorff; by compact space we shall always mean a space such that each open covering admits a finite subcovering. Quasitorus is a natural generalization of torus, an example, using the above notations, is \mathfrak{d}/Q .

Let us now go back to the problem posed above: let \mathfrak{d} be an n -dimensional real vector space, we consider a dimension n , nonrational, nonsimple convex polytope $\Delta \subset \mathfrak{d}^*$, together with the following data: a quasilattice Q in \mathfrak{d} and a set of generators of the 1-dimensional dual cones, contained in Q – in the rational case, given Δ in L , the generators are chosen to be primitive in L . We then consider the complex quasitorus $\mathfrak{d}_{\mathbb{C}}/Q$. We prove that, to each choice of generators and quasilattice Q , there corresponds a space X_{Δ} , acted on continuously by the quasitorus $\mathfrak{d}_{\mathbb{C}}/Q$. As in the rational case there is a one-to-one correspondence between p -dimensional orbits of the quasitorus and p -dimensional faces of the polytope. The space X_{Δ} is the compactification of the n -dimensional complex quasitorus $\mathfrak{d}_{\mathbb{C}}/Q$, obtained by gluing, to the open and dense orbit corresponding to the interior of the polytope, the smaller orbits corresponding to the other faces of the polytope. This defines a stratification of X_{Δ} : the orbits corresponding to the singular faces are the singular strata, they are complex quasifolds, isomorphic to $(\mathbb{C}^*)^p$ modulo the action of a discrete group; the maximal stratum, given by the union of the orbits corresponding to the regular faces, has the structure of an n -dimensional complex quasifold. We obtain in fact a *complex stratification by quasifolds*, that is, in a neighborhood of the stratum corresponding to a singular face F , the space X_{Δ} is biholomorphic to the twisted product, under the action of a discrete group, of the stratum itself by a *complex cone* over the space X_{Δ_F} ; when the polytope is rational we recover the stratified structure of the toric variety, since the twisting group is in this case finite. We therefore observe on the space X_{Δ} two different kinds of singularities: the stratified structure of X_{Δ} and the quasifold structure of the strata, due to nonsimpleness and nonrationality respectively. Let us see how these two features of Δ intervene in the quotient construction that produces our space X_{Δ} .

In [8] D. Cox constructs the toric variety corresponding to a rational convex polytope Δ in a lattice L as the categorical quotient of a suitable open subset C_{Δ}^d of \mathbb{C}^d , modulo the action of a subtorus of $(\mathbb{C}^*)^d$, where d is the number of facets of the polytope, or, in other terms, the number of 1-dimensional cones in the dual fan. The open subset C_{Δ}^d can be defined, in purely combinatorial terms, for any convex polytope. In order to

construct, in our setting, a suitable subgroup $N_{\mathbb{C}}$ of \mathbb{C}^d , we adopt the generalization of the Delzant procedure given in [18]. The group $N_{\mathbb{C}}$ thus obtained is the complexification of a nonclosed subgroup N of $(S^1)^d$, namely $N_{\mathbb{C}} = \exp(i\mathfrak{n})N$, where $\mathfrak{n} = \text{Lie}(N)$. In the simple case the orbits of $\exp(i\mathfrak{n})$ are closed; the geometric quotient, $\mathbb{C}_{\Delta}^d/N_{\mathbb{C}}$, gives a rationally smooth toric variety in the rational case and an n -dimensional compact complex quasifold in the nonrational case – this was proved, jointly with Elisa Prato, in [4]. But in the nonsimple case there are nonclosed $\exp(i\mathfrak{n})$ -orbits. To control their behavior we make use of suitable functions on \mathbb{C}^d of the kind

$$\sum_{k=1}^d |z_k|^{c_k},$$

$c_k \in \mathbb{R}$. These functions turn out to be an essential tool and play the role of the rational functions used by Cox in his paper. We are then able to generalize the notion of categorical quotient: the space X_{Δ} is finally defined to be the quotient $\mathbb{C}_{\Delta}^d/N_{\mathbb{C}}$. Remark that there are two distinct ways in which $N_{\mathbb{C}}$ -orbits are nonclosed, on one hand N itself is a nonclosed subgroup of $(S^1)^d$; on the other hand, since the polytope is nonsimple, there are nonclosed $\exp(i\mathfrak{n})$ -orbits: the two kind of singularities of X_{Δ} described above – quasifold structure of strata and stratification – are a direct consequence of this fact.

We continue by considering the symplectic quotient corresponding to Δ , with the same choice of generators and quasilattice. This is a space stratified by symplectic quasifolds [1]. We prove that X_{Δ} is homeomorphic to its symplectic counterpart, M_{Δ} , that such homeomorphism is compatible with the stratifications of X_{Δ} and M_{Δ} and its restriction to each stratum is a diffeomorphism, with respect to which the symplectic and complex structures of strata are compatible. In particular the space X_{Δ} is compact and its strata are in fact Kähler quasifolds.

In conclusion these results, which were announced in [2], enable us to associate a geometric space to an arbitrary convex polytope, generalizing the construction of a toric variety from a rational polytope; indeed, in the case of a rational polytope Δ in a lattice L , with the choice of primitive generators, our space X_{Δ} is the toric variety associated to the pair (Δ, L) . The geometry and topology of our spaces and the connection with the combinatorics of the associated polytopes are natural questions related to our work. An initial step towards a better understanding of these different aspects, that we are pursuing, is the study of cohomological invariants of our spaces. A first result in this direction can be found in [3].

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1 Singularity types

Before going into the core of the paper, we recall in this section the necessary notions, they can all be found in detail in the references.

1.1 Quasifolds

For the detailed definitions of real quasifolds and related geometrical objects we refer the reader to the original article by E. Prato [18] and to the Appendix of the joint paper with E. Prato [6], where some of the definitions were reformulated. The definition of complex quasifold can be derived naturally from that of real quasifold, a version based on [18] was given jointly with E. Prato in [4]. The definitions of quasitori and their Hamiltonian actions were introduced in [18], an extension of these notions to the complex set up was then given in [4].

Let us briefly recall the definition of complex quasifold. Let \tilde{V} be a complex quasifold and let Γ be a discrete group acting on \tilde{V} by biholomorphisms in such a way that the set of points where the action is not free, is closed and has minimal real codimension ≥ 2 . This implies that the action is free on an open, dense and connected set. The quotient space \tilde{V}/Γ is a *quasifold model*. Two models \tilde{V}/Γ and \tilde{W}/Δ are *biholomorphic* if there exists a homeomorphism $f: \tilde{V}/\Gamma \rightarrow \tilde{W}/\Delta$ that lifts to a biholomorphism $\tilde{f}: \tilde{V} \rightarrow \tilde{W}$. Let X be a topological space, a complex *quasifold chart* on X is an open subset $V \subset X$, homeomorphic to a model \tilde{V}/Γ . Given a pair of intersecting charts a suitable notion of holomorphic *change of charts* is defined, if satisfied the two charts are said to be *compatible*. A topological space is endowed with the structure of a *complex quasifold* if it is covered by a collection $\mathcal{A} = \{V_\alpha \simeq \tilde{V}_\alpha/\Gamma_\alpha \mid \alpha \in A\}$ of compatible complex charts.

Geometric objects on quasifolds, like differential forms, are defined on each \tilde{V}_α with the additional conditions that they descend to the quotient $\tilde{V}_\alpha/\Gamma_\alpha$ and that, when charts are overlapping, they glue suitably by means of the changes of charts.

As mentioned in the introduction quasilattices are very important:

Definition 1.1 (Quasilattice) *Let \mathfrak{d} be a real vector space of dimension n . A quasilattice Q in \mathfrak{d} is a \mathbb{Z} -submodule of \mathfrak{d} , generated by a set of generators of \mathfrak{d} .*

Notice that if $\text{rank}(Q) = n$ then Q is a lattice.

Definition 1.2 (Quasitorus, quasi-Lie algebra, exponential[18, 4]) *Let \mathfrak{d} be a vector space of dimension n and let $\mathfrak{d}_\mathbb{C} = \mathfrak{d} + i\mathfrak{d}$ its complexification. Let Q be a quasilattice in \mathfrak{d} . The quotient $D = \mathfrak{d}/Q$ (respect. $D_\mathbb{C} = \mathfrak{d}_\mathbb{C}/Q$) is an n -dimensional quasitorus (respect. complex quasitorus) with *quasi-Lie algebra* \mathfrak{d} (respect. $\mathfrak{d}_\mathbb{C}$). The quasitorus D (respect. $D_\mathbb{C}$) is a real (respect. complex) quasifold covered by one chart. The corresponding projection $\mathfrak{d} \rightarrow D$ (respect. $\mathfrak{d}_\mathbb{C} \rightarrow D_\mathbb{C}$) is the *exponential mapping* and we denote it by \exp (respect. $\exp_\mathbb{C}$).*

Notice that when Q is a true lattice then the quasitorus \mathfrak{d}/Q is a torus.

Example 1.3 Consider the quasilattice $\mathbb{Z} + \alpha\mathbb{Z}$ in \mathbb{R} , with $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. A basic example of real quasifold is E. Prato's quasicircle: the real quasitorus $D_\alpha^1 = \mathbb{R}/\mathbb{Z} + \alpha\mathbb{Z}$ [18]. Notice that, if α is taken in \mathbb{Q} , then D_α^1 is either an orbifold or S^1 . A basic example of complex quasifold is the complexification $(D_\alpha^1)_\mathbb{C}$ of D_α^1 , this is the complex quasitorus given by $\mathbb{C}/(\mathbb{Z} + \alpha\mathbb{Z})$.

1.2 Complex stratifications

We recall the definition of decomposition and stratification by quasifolds (cf. [1]).

Definition 1.4 Let X be a topological space. A *decomposition of X by quasifolds* is a collection of disjoint, locally closed, connected quasifolds \mathcal{T}_F ($F \in \mathcal{F}$), called *pieces*, such that

1. The set \mathcal{F} is finite, partially ordered and has a maximal element;
2. $X = \bigcup_F \mathcal{T}_F$;
3. $\mathcal{T}_F \cap \overline{\mathcal{T}_{F'}} \neq \emptyset$ iff $\mathcal{T}_F \subseteq \overline{\mathcal{T}_{F'}}$ iff $F \leq F'$;
4. the piece corresponding to the maximal element is open and dense in X .

The space X is then said to be an *m -dimensional space decomposed by quasifolds*, where m is the dimension of the maximal piece. We call the maximal piece the regular piece and the other pieces singular.

A mapping from a decomposed space X to a decomposed space X' is smooth (respect. a diffeomorphism) if it is a continuous mapping (respect. a homeomorphism) that respects the decomposition and is smooth (respect. a diffeomorphism) when restricted to pieces.

A stratification is a decomposition that satisfies a local triviality condition.

Let L be a space decomposed by quasifolds. A *cone over L* , denoted by $C(L)$, is the space $[0, 1] \times L / \sim$, where two points (t, l) and (t', l') in $[0, 1] \times L$ are equivalent if and only if $t = t' = 0$. The decomposition of L induces a decomposition of the cone. Now let t be a point in a quasifold \mathcal{T} and let $B \simeq \tilde{B}/\Gamma$ a local model of \mathcal{T} containing t . The decomposition of L induces a decomposition of the product $\tilde{B} \times C(L)$. Suppose, in addition, that Γ acts freely on \tilde{B} and that the space L is endowed with an action of Γ that preserves the decomposition; then the product $\tilde{B} \times C(L)$ is acted on by Γ and the quotient $(\tilde{B} \times C(L))/\Gamma$ inherits the decomposition of $\tilde{B} \times C(L)$. Moreover the quotient $(\tilde{B} \times C(L))/\Gamma$ fibers over B with fiber $C(L)$.

Definition 1.5 Let X be an m -dimensional space decomposed by real quasifolds, the decomposition of X is said to be a *stratification by quasifolds* if each singular piece \mathcal{T} , called *stratum*, satisfies the following conditions:

1. let r be the real dimension of \mathcal{T} , for every point $t \in \mathcal{T}$ there exist: an open neighborhood U of t in X ; a local model $B \simeq \tilde{B}/\Gamma$ in \mathcal{T} containing t and such that Γ acts freely on \tilde{B} ; a $(m - r - 1)$ -dimensional compact space L , called the *link* of t , decomposed by quasifolds; an action of the group Γ on L , preserving the decomposition of L and such that the pieces of the induced decomposition of $\tilde{B} \times C(L)/\Gamma$ are quasifolds; finally a homeomorphism $h: (\tilde{B} \times C(L))/\Gamma \rightarrow U$ that respects the decompositions and takes each piece of $(\tilde{B} \times C(L))/\Gamma$ diffeomorphically into the corresponding piece of U ;
2. the decomposition of the link L satisfies condition 1.

The definition is recursive and, since the dimension of L decreases at each step, we end up, after a finite number of steps, with links that are compact quasifolds.

Remark 1.6 Notice that, if the discrete groups Γ 's are finite for any possible F , $t \in \mathcal{T}_F$ and B , then the twisted products $\tilde{B} \times \mathbb{C}(L)/\Gamma$ become trivial and the singular strata turn out to be smooth manifolds, since Γ 's act freely. Therefore our stratification satisfies in this case the local triviality condition of the classical definition of stratification, moreover, strata are smooth, with the only possible exception of the principal stratum, that might be an orbifold.

We can then ask that the stratification be endowed with a global complex structure [2], namely strata are complex quasifolds and the complex structure of each stratum is compatible with the stratification, these are very strong requirements that are not usually satisfied, for example complex stratifications are usually far from being holomorphically locally trivial, however, toric varieties and our toric spaces satisfy the following definition:

Definition 1.7 (Complex stratification) The stratified space X is endowed with a complex structure if the following conditions are satisfied:

1. for each link L there exist: a compact space Y stratified by quasifolds; a smooth surjective map $s : L \rightarrow Y$; a 1-parameter subgroup S of a real torus, acting smoothly on L , with 0-dimensional stabilizer, such that, for each $y \in Y$, the fiber $s^{-1}(y)$ is diffeomorphic to the quotient of S by the stabilizer of S on the fiber itself;
2. each piece of the stratified spaces X , $C(L)$'s and Y 's is endowed with a complex structure;
3. the natural projection $C(L) \setminus \{\text{cone pt}\} \rightarrow Y$, induced by s , when restricted to each piece, is holomorphic, with fiber over each $y \in Y$ biholomorphic to the quotient of $S_{\mathbb{C}}$ by the stabilizer on the fibre itself;
4. the space X is locally biholomorphic to the product $\tilde{B} \times C(L)/\Gamma$, that is the identification mapping h is a biholomorphism when restricted to pieces. We call Y *complex link* and $C(L)$ *complex cone* over Y .

2 The construction of the quotient

In this section we describe the subsequent steps that lead to the construction of the quotient space associated to a convex polytope Δ .

2.1 The open subset \mathbb{C}_{Δ}^d in \mathbb{C}^d

Let \mathfrak{d} be a real vector space of dimension n , and let Δ be a convex polytope of dimension n in the dual space \mathfrak{d}^* . Let d be the number of facets of Δ , write Δ as intersection of half spaces

$$\Delta = \bigcap_{j=1}^d \{ \mu \in \mathfrak{d}^* \mid \langle \mu, X_j \rangle \geq \lambda_j \} \quad (1)$$

where X_1, \dots, X_d are the chosen generators of the 1-dimensional cones of the fan dual to Δ ; the coefficients λ_j 's are uniquely determined by the X_j 's. For each open face F of Δ we denote by I_F the subset of $\{1, \dots, d\}$ such that

$$F = \{ \mu \in \Delta \mid \langle \mu, X_j \rangle = \lambda_j \text{ if and only if } j \in I_F \}. \quad (2)$$

The n -dimensional open face of Δ corresponds to the empty set. A partial order on the set of all faces of Δ is defined by setting $F \leq F'$ (we say F contained in F') if $F \subseteq \overline{F'}$. The polytope Δ is the disjoint union of its faces. Let $r_F = \text{card}(I_F)$; we have the following definitions:

Definition 2.1 A p -dimensional face F of the polytope is said to be *singular* if $r_F > n - p$, *regular* if $r_F = n - p$.

Remark 2.2 Let F be a p -dimensional singular face in \mathfrak{d}^* , then $p < n - 2$. For example: a polytope in $(\mathbb{R}^2)^*$ is simple; the singular faces of a nonsimple polytope in $(\mathbb{R}^3)^*$ are 0-dimensional.

Now let \mathbb{K} be one of the following sets $\mathbb{C}, \mathbb{C}^*, \mathbb{R}, \mathbb{R}^*$, all of them considered naturally immersed in \mathbb{C} . Let J be a subset of $\{1, \dots, d\}$ and let J^c be its complement. We denote by

$$\mathbb{K}^J = \{ (z_1, \dots, z_d) \in \mathbb{C}^d \mid z_j \in \mathbb{K} \text{ if } j \in J, z_j = 0 \text{ if } j \notin J \}. \quad (3)$$

We have $\mathbb{K}^d = \mathbb{K}^J \times (\mathbb{K})^{J^c}$. Let $\underline{z} \in \mathbb{K}^d$, we denote by \underline{z}_J its projection onto the factor \mathbb{K}^J . By T^J we denote the subtorus $\{ (t_1, \dots, t_d) \in T^d \mid t_j = 1 \text{ if } j \notin J \}$. Let F be a p -dimensional face and let I_F be the corresponding set of indices. To lighten the notation we shall omit the I and simply write $\mathbb{K}^F, \mathbb{K}^{F^c}, T^F, \dots$, instead of $\mathbb{K}^{I_F}, \mathbb{K}^{I_F^c}, T^{I_F}, \dots$.

Let us denote by \mathbb{C}_Δ^d the open subset of \mathbb{C}^d given by

$$\mathbb{C}_\Delta^d = \cup_{F \in \Delta} \mathbb{C}^F \times \mathbb{C}^{*F^c} \quad (4)$$

Notice that in the definition of the open subset \mathbb{C}_Δ^d only the combinatorics of the polytope intervenes. Moreover, the open subset \mathbb{C}_Δ^d coincides with the one defined in [8] for the rational case.

2.2 The group $N_{\mathbb{C}}$

It is in the definition of the group acting on \mathbb{C}_Δ^d that nonrationality comes in. In order to obtain the group we adopt a generalization to nonrational polytopes [18] of the Delzant procedure [9]. Let Q be a quasilattice in the space \mathfrak{d} containing the elements X_j (for example $\text{Span}_{\mathbb{Z}}\{X_1, \dots, X_d\}$) and let $\{e_1, \dots, e_d\}$ denote the standard basis of \mathbb{R}^d ; consider the surjective linear mappings

$$\begin{array}{ccc} \pi: & \mathbb{R}^d & \longrightarrow & \mathfrak{d} \\ & e_j & \longmapsto & X_j, \end{array} \quad \begin{array}{ccc} \pi_{\mathbb{C}}: & \mathbb{C}^d & \longrightarrow & \mathfrak{d}_{\mathbb{C}} \\ & e_j & \longmapsto & X_j, \end{array} \quad (5)$$

when no ambiguity can arise we shall drop the subscript \mathbb{C} . Consider the quasitorus \mathfrak{d}/Q and its complexification $\mathfrak{d}_{\mathbb{C}}/Q$. Each of the mappings π and $\pi_{\mathbb{C}}$ induces a group homomorphism,

$$\Pi: T^d = \mathbb{R}^d/\mathbb{Z}^d \longrightarrow \mathfrak{d}/Q$$

and

$$\Pi_{\mathbb{C}}: T_{\mathbb{C}}^d = \mathbb{C}^d/\mathbb{Z}^d \longrightarrow \mathfrak{d}_{\mathbb{C}}/Q.$$

We define N to be the kernel of the mapping Π and $N_{\mathbb{C}}$ to be the kernel of the mapping $\Pi_{\mathbb{C}}$. The mapping $\Pi_{\mathbb{C}}$ defines the isomorphism

$$T_{\mathbb{C}}^d/N_{\mathbb{C}} \longrightarrow \mathfrak{d}_{\mathbb{C}}/Q \quad (6)$$

Remark 2.3 As in the simple case [4], we have, for the complexified group $N_{\mathbb{C}}$, the polar decomposition, namely:

$$N_{\mathbb{C}} = NA, \quad (7)$$

where $A = \exp(i\mathfrak{n})$: every element $w \in N_{\mathbb{C}}$ can be written uniquely as $x \exp(iY)$ where $x \in N$ and $Y \in \mathfrak{n}$. This follows from the definition of N and $N_{\mathbb{C}}$, indeed $N_{\mathbb{C}} = \{\exp(Z) \mid Z \in \mathbb{C}^d \text{ and } \pi_{\mathbb{C}}(Z) \in Q\}$. Write $Z = X + iY$, then $\pi_{\mathbb{C}}(Z) \in Q$ if and only if $\pi(X) \in Q$ and $\pi(Y) = 0$, which implies (7).

Notice that neither N nor $N_{\mathbb{C}}$ is a torus unless Q is a lattice.

The next statement will be of great help when analysing the local structure of our space. Define \mathcal{I} to be the set of subsets I of $\{1, \dots, d\}$ such that $\{X_j \mid j \in I\}$ is a basis of \mathfrak{d} and $I \subset I_{\mu}$ for a vertex μ of Δ . If a vertex μ is non singular then I_{μ} itself is in \mathcal{I} . Define the group

$$\Gamma_I = N \cap T^I,$$

observe that Γ_I is discrete by (9). We are now able to state a Lemma, which was proved in the simple case in [4, Lemma 2.3]. The proof goes through with no changes, we briefly recall it for completeness.

Lemma 2.4 *Let $I \in \mathcal{I}$. Then we have that*

1. $T_{\mathbb{C}}^d/T_{\mathbb{C}}^I \simeq N_{\mathbb{C}}/\Gamma_I$;
2. $N_{\mathbb{C}} = \Gamma_I \exp(\mathfrak{n} + i\mathfrak{n})$;
3. *given any complement $\mathfrak{b}_{\mathbb{C}}$ of \mathbb{C}^I in \mathbb{C}^d , we have that*

$$\mathfrak{n}_{\mathbb{C}} = \{Z - \pi_I^{-1}(\pi(Z)) \mid Z \in \mathfrak{b}_{\mathbb{C}}\}.$$

Proof. 1. The natural group homomorphism $N_{\mathbb{C}} \longrightarrow T_{\mathbb{C}}^d/T_{\mathbb{C}}^I$ induces an isomorphism $T_{\mathbb{C}}^d/T_{\mathbb{C}}^I \simeq N_{\mathbb{C}}/\Gamma_I$.

2. Every element in $N_{\mathbb{C}}$ can be written in the form $\exp(Z)$, where $Z \in \mathbb{C}^d$ is such that $\pi(Z) \in Q$. Then $Z - \pi_I^{-1}(\pi(Z)) \in \mathfrak{n}_{\mathbb{C}}$, and $\exp(\pi_I^{-1}(\pi(Z))) \in \Gamma_I$. The group $\Gamma_I \cap \exp(\mathfrak{n}_{\mathbb{C}})$ is not necessarily trivial, so the decomposition need not to be unique.

3. Split $V \in \mathfrak{n}_{\mathbb{C}}$ as $V = Z_1 + Z_2$ according to $\mathbb{C}^d = \mathbb{C}^I \oplus \mathfrak{b}_{\mathbb{C}}$, then $\pi(V) = 0$ implies that $Z_1 = -\pi_I^{-1}(\pi(Z_2))$. \square The following corollary is important, it means that the action of N on \mathbb{C}^d is not far from being proper:

Corollary 2.5 *Let $I \in \mathcal{I}$ and let $\{c_m\} \in N$ be a sequence, then there exists a subsequence $\{c_k\}$ such that for each k the element $c_k = \gamma_k b_k$ with $\gamma_k \in \Gamma_I$, $b_k \in N$ and the sequence $\{b_k\}$ is converging to an element $b \in N$.*

Proof. By Lemma 2.4 the sequence c_m can be written as

$$c_m = \gamma'_m \left(\exp(-\pi_I^{-1}(\pi(Y_m))) \exp(Y_m) \right),$$

with $Y_m \in \mathbb{R}^{I^c}$, therefore there exists a subsequence $Y'_k \in \mathbb{R}^{I^c}$ such that $Y_k - Y'_k \in \mathbb{Z}^{I^c}$ and Y'_k is convergent in \mathbb{R}^{I^c} . Therefore $c_k = \gamma_k b_k$ with $\gamma_k \in \Gamma_I$ and $b_k = \exp(-\pi_I^{-1}(\pi(Y_k))) \exp(Y_k)$ convergent in N by continuity. \square

2.3 The $\exp(in)$ -orbits

Let \underline{z} be a point in \mathbb{C}_Δ^d , we say that the A -orbit $A\underline{z}$ is closed if it is closed in \mathbb{C}_Δ^d . The first step towards the construction of our toric space as quotient is given by the following theorem:

Theorem 2.6 (Closed orbits) *Let $\underline{z} \in \mathbb{C}_\Delta^d$. Then the A -orbit through \underline{z} , $A\underline{z}$, is closed if and only if there exists a face F such that \underline{z} is in $(\mathbb{C}^*)^{F^c}$. Moreover, if $A\underline{z}$ is nonclosed, then its closure contains one and only one closed A -orbit.*

Proof. Working on the proof of [8, Thm 2.1] did help us to find an argument for the proof of the first part of our statement. Let $\underline{z} \in (\mathbb{C}^*)^{F^c}$. We want to prove that $A\underline{z}$ is closed in \mathbb{C}_Δ^d . Observe first that $A\underline{z} \subset (\mathbb{C}^*)^{F^c}$. Since $\mathbb{C}_\Delta^d = \cup_G (\mathbb{C}^G \times (\mathbb{C}^*)^{G^c})$, it suffices to prove that $A\underline{z}$ is closed in the open subset $\mathbb{C}^G \times (\mathbb{C}^*)^{G^c}$ for each face G of Δ such that $A\underline{z} \subset \mathbb{C}^G \times (\mathbb{C}^*)^{G^c}$, namely for each face G such that $I_F \subseteq I_G$, that is such that $G \subseteq F$. Let us consider first the case of G properly contained in F . Let $\xi \in F \setminus G$ and $\eta \in G$: the coefficients

$$c_j = \langle \xi, X_j \rangle - \langle \eta, X_j \rangle$$

have the following properties:

$$\begin{aligned} c_j &= \lambda_j - \lambda_j = 0, & \text{for } j \in I_F \\ c_j &= \langle \xi, X_j \rangle - \lambda_j > 0 & \text{for } j \in I_G \setminus I_F. \end{aligned}$$

We define on the subset $\mathbb{C}^G \times (\mathbb{C}^*)^{G^c}$ the continuous function

$$P(\underline{w}) = \prod_{j=1}^d |w_j|^{c_j} \tag{8}$$

The function P is well defined. We prove that P is invariant under the action of $N_\mathbb{C}$. The function P is clearly invariant under the action of N , we need to prove that it is invariant under the action of $A = \exp(in)$. Consider the exact sequence:

$$0 \longrightarrow \mathfrak{n} \xrightarrow{\iota} \mathbb{R}^d \xrightarrow{\pi} \mathfrak{d} \longrightarrow 0 \tag{9}$$

and the dual sequence

$$0 \longrightarrow \mathfrak{d}^* \xrightarrow{\pi^*} (\mathbb{R}^d)^* \xrightarrow{\iota^*} \mathfrak{n}^* \longrightarrow 0 \tag{10}$$

Let $X \in \mathfrak{n}$ and $a = \exp(i\iota(X))$, then

$$a\underline{w} = (e^{-2\pi\langle\iota(X), e_1^*\rangle} w_1, \dots, e^{-2\pi\langle\iota(X), e_d^*\rangle} w_d)$$

and

$$P(a\underline{w}) = e^{-2\pi \sum_{j=1}^d (\langle\xi, X_j\rangle - \langle\eta, X_j\rangle) \langle\iota(X), e_j^*\rangle} \prod_{k=1}^d |w_k|^{c_k}$$

Notice that

$$\sum_{j=1}^d (\langle\xi, X_j\rangle - \langle\eta, X_j\rangle) \langle\iota(X), e_j^*\rangle = \langle X, \iota^* \left(\sum_{j=1}^d \langle\pi^*(\xi - \eta), e_j\rangle e_j^* \right) \rangle.$$

Therefore, since $\sum_{j=1}^d \langle\pi^*(\xi - \eta), e_j\rangle e_j^* \in \text{Im } \pi^*$, by (10) we obtain

$$i^* \left(\sum_{j=1}^d \langle\pi^*(\xi - \eta), e_j\rangle e_j^* \right) = 0$$

and therefore

$$P(a\underline{w}) = P(\underline{w}).$$

Remark that $P(\underline{z}) \neq 0$ and define the subset $P_{\underline{z}} = \{\underline{w} \in \mathbb{C}^G \times (\mathbb{C}^*)^{G^c} \mid P(\underline{w}) = P(\underline{z})\}$. The set $P_{\underline{z}}$ is closed, is contained in $\mathbb{C}^F \times (\mathbb{C}^*)^{\bar{F}^c}$ and the orbit $A\underline{z}$ is contained in $P_{\underline{z}}$, since the function P is A -invariant. It follows that $\overline{A\underline{z}}$ is contained in $P_{\underline{z}}$ and therefore in $\mathbb{C}^F \times (\mathbb{C}^*)^{F^c}$. We prove that this implies

$$\overline{A\underline{z}} = A\underline{z}.$$

Let \underline{w} be a point in the closure of $A\underline{z}$ in $\mathbb{C}^G \times (\mathbb{C}^*)^{G^c}$, then $\underline{w} \in \mathbb{C}^F \times (\mathbb{C}^*)^{F^c}$ and there exists a sequence $Y_n \in \mathfrak{n}$ such that the sequence $\exp(iY_n)\underline{z}$ converges to \underline{w} . Therefore, for $j \notin I_F$,

$$\lim_{n \rightarrow +\infty} e^{-2\pi\langle\iota(Y_n), e_j^*\rangle} z_j = w_j.$$

This implies that there exists $Y' \in \mathbb{R}^{F^c}$ such that

$$\lim_{n \rightarrow +\infty} e^{-2\pi\langle\iota(Y_n), e_j^*\rangle} = e^{-2\pi\langle Y', e_j^*\rangle}$$

and

$$w_j = e^{-2\pi\langle Y', e_j^*\rangle} z_j.$$

Now remark that $(\pi \circ \iota)(Y_n) = 0$, which gives

$$\sum_{j \in I_F} \langle\iota(Y_n), e_j^*\rangle X_j = - \sum_{j \notin I_F} \langle\iota(Y_n), e_j^*\rangle X_j. \quad (11)$$

Let

$$\mathfrak{d}_F = \text{Span}\{X_j \mid j \in F\}.$$

The sequence $\sum_{j \in I_F} \langle\iota(Y_n), e_j^*\rangle X_j$ is in \mathfrak{d}_F , by (11) it converges to $-\sum_{j \notin I_F} \langle Y', e_j^*\rangle X_j$, which must therefore lie in \mathfrak{d}_F . Let $Y'' \in \mathbb{R}^F$ such that

$$\sum_{j \in I_F} \langle Y'', e_j^*\rangle X_j = - \sum_{j \notin I_F} \langle Y', e_j^*\rangle X_j,$$

and set $Y = Y' + Y'' \in \mathbb{R}^d$. It follows that $\pi(Y) = 0$, therefore $Y \in \mathfrak{n}$ and

$$\underline{w} = \exp iY \underline{z}.$$

The above argument also proves that $A\underline{z}$ is closed in $\mathbb{C}^F \times (\mathbb{C}^*)^{F^c}$.

Now consider a point \underline{z} such that $\underline{z} \notin (\mathbb{C}^*)^{G^c}$ for any face G of Δ . Here suggestions for the proof come from an argument by I. Musson [16] used by Cox in his proof. Take a face F such that

$$\underline{z} \in \mathbb{C}^F \times (\mathbb{C}^*)^{F^c}.$$

Then F must be singular, otherwise we would have \underline{z} contained in $(\mathbb{C}^*)^{G^c}$ for some face G containing F . Therefore there exist coefficients y_k not all zero such that

$$\sum_{k \in I_F} y_k X_k = 0.$$

We want to prove that the coefficients y_k can be chosen so that there exists a $j \in I_F$ with the property that:

$$z_j \neq 0, \quad y_j > 0$$

Let $I_{\underline{z}} = \{k \in I_F \mid z_k = 0\}$ and take

$$E = \cap_{k \in I_{\underline{z}}} \{\xi \in \Delta \mid \langle \xi, X_k \rangle = \lambda_k\}, \quad (12)$$

then either $E = F$ or $F \subset E$: if $E = F$ then $\{X_k \mid k \in I_{\underline{z}}\}$ span \mathfrak{d}_F , therefore we can find an index j with the required properties; if $F \subset E$ then any $j \in I_E \setminus I_{\underline{z}}$, which is nonempty by our hypothesis on \underline{z} , satisfies the required properties. Let y_k be the chosen coefficients for $k \in I_F$, set $y_k = 0$ for $k \notin I_F$ and let

$$Y = (y_1, \dots, y_d),$$

since $\pi(Y) = 0$ we have that $Y \in \mathfrak{n}$. Remark now that

$$\lim_{t \rightarrow +\infty} \exp(itY) \underline{z} \notin A\underline{z},$$

the orbit is therefore nonclosed.

Now we want to prove that $\overline{A\underline{z}}$ contains one and only one closed orbit. Let E be the face defined by (12), then $I_{\underline{z}} \subset I_E$, moreover $\{X_k \mid k \in I_{\underline{z}}\}$ span \mathfrak{d}_E . For each $j \in I_E \setminus I_{\underline{z}}$ we can therefore find $y_k^j \in \mathbb{R}$ such that:

$$\begin{aligned} y_j^j &= 1 \\ y_k^j &= 0, \quad k \notin I_{\underline{z}}, \quad k \neq j \\ \sum_{k=1}^d y_k^j X_k &= 0 \end{aligned}$$

Let $Y^j = (y_1^j, \dots, y_d^j)$, we have that $\pi(Y^j) = 0$, therefore $Y^j \in \mathfrak{n}$. Therefore

$$\lim_{t \rightarrow +\infty} \left(\prod_{j \in I_E \setminus I_{\underline{z}}} \exp(itY^j) \right) \underline{z} = \underline{z}_{E^c}$$

implies that the closure of the orbit $A\underline{z}$ contains the orbit $A\underline{z}_{E^c}$, which is closed.

Suppose now that there is another closed orbit, $A\underline{u}$, in $\overline{A\underline{z}}$. Then we have $\underline{u} \in \mathbb{C}^{G^c}$ for some face G such that $F \subseteq G$. If $E \neq G$ we can take $\xi \in E \setminus E \cap G$ and $\eta \in G \setminus E \cap G$. We can then construct a function P as in (8) with coefficients

$$c_j = \langle \xi, X_j \rangle - \langle \eta, X_j \rangle.$$

We have

$$c_j = \lambda_j - \lambda_j = 0, \quad \text{for } j \in I_E \cap I_G$$

$$c_j = \langle \xi, X_j \rangle - \lambda_j > 0 \quad \text{for } j \in I_G \setminus (I_E \cap I_G).$$

$$c_j = \lambda_j - \langle \eta, X_j \rangle < 0 \quad \text{for } j \in I_E \setminus (I_E \cap I_G).$$

Remark that $I_{\underline{z}} \subset I_E \cap I_G$. The function P is well defined on the orbits $A\underline{z}$ and $A\underline{u}$, it is A -invariant and therefore constant and nonzero on the A -orbit $A\underline{z}$. This implies that $I_G = I_G \cap I_E$. Taking P^{-1} proves that $I_E = I_G \cap I_E$. It follows that $E = G$.

In order to procede we rely on the following statement that will be proved later on, in Corollary 3.10.

Let G be a face of Δ and let $\underline{w} \in (\mathbb{C}^)^G$, then there exist a unique point \underline{w}^0 in $A\underline{w}$ and a unique point $\xi \in G$ such that*

$$\langle \xi, X_j \rangle - \lambda_j = |w_j^0|^2, \quad j = 1, \dots, d$$

Now let $A\underline{w}$ and $A\underline{u}$ be two closed orbits in $\overline{A\underline{z}}$ and let (\underline{w}^0, ξ) and (\underline{u}^0, η) as in the statement right above. Suppose that the two orbits do not coincide, therefore $(\underline{w}, \xi) \neq (\underline{u}, \eta)$. Let P be the function given in (8), with coefficients $c_j = \langle \xi, X_j \rangle - \langle \eta, X_j \rangle$. The function P is well is constant and nonzero on $A\underline{z}$ and hence by continuity on the orbits $A\underline{u}$ and $A\underline{w}$. It follows that $P(\underline{u}^0)/P(\underline{w}^0) = 1$. Now let $J^+ = \{j \mid c_j > 0\}$ and $J^- = \{j \mid c_j < 0\}$, we have

$$1 = P(\underline{u}^0)/P(\underline{w}^0) = (\prod_{k \in J^+} |u_k^0/w_k^0|^{c_k}) (\prod_{k \in J^-} |u_k^0/w_k^0|^{c_k})$$

now notice that

$$c_j = |w_j^0|^2 - |u_j^0|^2$$

which leads to a contradiction, therefore $J^+ = J^- = \emptyset$ and $|w_j^0|^2 = |u_j^0|^2$ for all $j \in \{1, \dots, d\}$, which implies $\underline{w}^0 = \underline{u}^0$, since they lie in the closure of the same A -orbit. \square

2.4 The quotient by the nonclosed group $N_{\mathbb{C}}$

Theorem 2.6 allows us to define on the open set \mathbb{C}_{Δ}^d the following equivalence relation: two points \underline{z} and \underline{w} are equivalent with respect to the action of the group $N_{\mathbb{C}}$,

$$\underline{z} \sim_N \underline{w}, \tag{13}$$

if and only if

$$\left(N(\overline{A\underline{z}})\right) \cap \left(\overline{A\underline{w}}\right) \neq \emptyset, \tag{14}$$

where the closure is meant in \mathbb{C}_{Δ}^d .

Proposition 2.7 *The relation defined by (14) is an equivalence relation.*

Proof. The relation defined in (14) is symmetric: let \underline{z} and \underline{w} such that $\underline{z} \sim_N \underline{w}$, then $(N(\overline{A\underline{z}})) \cap (\overline{A\underline{w}}) \neq \emptyset$. Remark first that $\overline{A\underline{z}}$ is a union of A -orbits and that $N(\overline{A\underline{z}})$ is also a union of A -orbits: for the first claim take $\underline{u} \in \overline{A\underline{z}}$, then there exists a sequence $a_k \in A$ such that $a_k \underline{z} \rightarrow \underline{u}$, then, since the action is continuous, for each $a \in A$ $a_k a \underline{z} \rightarrow a \underline{u}$, therefore $A\underline{u} \subset \overline{A\underline{z}}$; for the second claim take $\underline{u} \in N(\overline{A\underline{z}})$ then there exists $c \in N$ and $\underline{u}' \in \overline{A\underline{z}}$ such that $\underline{u} = c\underline{u}'$, but, for what we have just observed, the whole orbit $A\underline{u}'$ is contained in $\overline{A\underline{z}}$, therefore the whole orbit $A\underline{u}$ is contained in $N(\overline{A\underline{z}})$. It follows that the intersection $(N(\overline{A\underline{z}})) \cap (\overline{A\underline{w}})$ is a union of A -orbits. Let $A\underline{u} \subset (N(\overline{A\underline{z}})) \cap (\overline{A\underline{w}})$, then there exists $c \in N$ such that $A\underline{u}$ is an A -orbit in $c(\overline{A\underline{z}})$, therefore $A(c^{-1}\underline{u})$ is an A -orbit in $\overline{A\underline{z}} \cap c(\overline{A\underline{w}})$, hence $(N(\overline{A\underline{w}})) \cap (\overline{A\underline{z}}) \neq \emptyset$. This proves that the relation is symmetric. We prove now that the relation is transitive. Remark first that if $A\underline{u}$ is in the intersection $(N(\overline{A\underline{w}})) \cap (\overline{A\underline{z}}) \neq \emptyset$, then its closure also lies in the intersection, by Theorem 2.6 $\overline{A\underline{u}}$ contains one and only one closed A -orbit. Let $\underline{z} \sim_N \underline{v}$ and $\underline{v} \sim_N \underline{w}$. The closed orbit in $N(\overline{A\underline{z}}) \cap \overline{A\underline{v}}$ and the closed orbit in $N(\overline{A\underline{v}}) \cap \overline{A\underline{w}}$ coincide since they both lie in $\overline{A\underline{v}}$, therefore $\underline{z} \sim_N \underline{w}$. \square We define the space X_Δ to be the quotient of \mathbb{C}_Δ^d by the equivalence relation just defined, we denote the quotient by

$$X_\Delta = \mathbb{C}_\Delta^d // N_{\mathbb{C}}.$$

Notice that, if the polytope is simple, then $\mathbb{C}_\Delta^d = \cup_{F \in \Delta} (\mathbb{C}^*)^F$ and the quotient X_Δ is just the orbit space endowed with the quotient topology.

Remark 2.8 The construction of the quotient can be carried out for any convex polytope, rational or not. If we start with a rational polytope in a lattice L and we choose the primitive generators of the 1-dimensional cone in the dual fan, then the above quotient is the toric variety associated to the polytope in L [8]. In general there are many toric spaces associated to a given convex polytope, depending on the choice of quasilattice and generators. See the model example of the unit interval and the family of toric spaces associated to it [18]. There are applications in which it is natural to consider a rational polytope in a quasilattice [5, 7].

3 The structure of the quotient X_Δ

3.1 The decomposition and the structure of the pieces

The decomposition in pieces of the quotient X_Δ reflects the geometry of the polytope Δ . The indexing set \mathcal{F} for the decomposition of X_Δ is given by the set of the singular faces of the polytope Δ , with the partial order defined in Section 2.1, with the addition of a maximal element. The maximal piece is

$$\mathcal{T}_{\max} = \cup_{F \text{ reg}} (\mathbb{C}^F \times (\mathbb{C}^*)^{F^c}) / N_{\mathbb{C}} = \cup_{F \text{ reg}} (\mathbb{C}^*)^{F^c} / N_{\mathbb{C}}.$$

Then there is a piece \mathcal{T}_F for each singular face F of Δ :

$$\mathcal{T}_F = \{z \in \mathbb{C}_\Delta^d \mid \overline{A\underline{z}} \cap (\mathbb{C}^*)^{F^c} \neq \emptyset\} // N_{\mathbb{C}}.$$

Theorem 2.6 implies that the space X_Δ is given by the union of the regular and singular pieces. The structure of X_Δ as decomposed space and the properties that characterize X_Δ as a toric space associated to Δ are described in this section:

Theorem 3.1 (Quasifold structure of strata) *The subset \mathcal{T}_F of X_Δ corresponding to a p -dimensional singular face of Δ is a p -dimensional complex quasifold. The subset \mathcal{T}_{\max} is an n -dimensional complex quasifold. These subsets give a decomposition by complex quasifolds of X_Δ .*

Proof. *Regular piece* Consider the regular piece \mathcal{T}_{\max} . The proof that it is an n -dimensional complex quasifold goes very similarly to the proof, given in [4, Thm2.2], that the space corresponding to a simple polytope, with a given choice of normals and quasilattice, is an n -dimensional complex quasifold. Notice that $\cup_{F \text{ reg}} (\mathbb{C}^*)^{F^c}$ is covered by the sets $\hat{V}_I = (\mathbb{C}^I \times (\mathbb{C}^*)^{I^c}) \cap (\cup_{F \text{ reg}} (\mathbb{C}^*)^{F^c})$, with $I \in \mathcal{I}$. Let $\tilde{V}_I \subset \mathbb{C}^I$ be the image of the natural projection mapping from $(\mathbb{C}^I \times (\mathbb{C}^*)^{I^c}) \cap (\cup_{F \text{ reg}} (\mathbb{C}^*)^{F^c})$ to \mathbb{C}^I . Consider the mapping

$$p_I : \tilde{V}_I \longrightarrow \hat{V}_I$$

defined by

$$(p_I(\underline{z}))_j = \begin{cases} z_j & \text{if } j \in I \\ 1 & \text{if } j \in I^c \end{cases}$$

The mapping p_I induces a homeomorphism

$$\phi_I : \tilde{V}_I / \Gamma_I \longrightarrow \hat{V}_I / N_{\mathbb{C}}.$$

The proof that ϕ_I is a homeomorphism, as well as the proof that the charts $(\hat{V}_I / N_{\mathbb{C}}, \phi_I, \tilde{V}_I / \Gamma_I)$ give an atlas of the regular piece, go along the lines of the proof of [4, Thm.2.2], we will not repeat the argument here.

Singular pieces Let F be a singular face, we want to characterize those points \underline{z} which are equivalent, with respect to (14), to points in $(\mathbb{C}^*)^{F^c}$. Let I_h , with $h = 1, \dots, m_F$, be the subsets of I_F such that

$$F = \cap_{j \in I_h} \{\xi \in \Delta \mid \langle \xi, X_j \rangle = \lambda_j\}$$

From the proof of Theorem 2.6 it follows that

$$\{\underline{z} \in \mathbb{C}_\Delta^d \mid \overline{A\underline{z}} \cap (\mathbb{C}^*)^{F^c} \neq \emptyset\} = \cup_{h=1}^{m_F} (\mathbb{C}^*)^{I_h^c}.$$

Moreover, if we set $\pi_h : (\mathbb{C}^*)^{I_h^c} \longrightarrow (\mathbb{C}^*)^{F^c}$, for a given $N_{\mathbb{C}}$ -invariant subset U of $(\mathbb{C}^*)^{F^c}$ we have that:

$$\{\underline{z} \in \mathbb{C}_\Delta^d \mid \overline{A\underline{z}} \cap U \neq \emptyset\} = \cup_{h=1}^{m_F} \pi_h^{-1}(U). \quad (15)$$

This, together with Theorem 2.6, implies that the inclusion mapping

$$(\mathbb{C}^*)^{F^c} \hookrightarrow \cup_{h=1}^{m_F} (\mathbb{C}^*)^{I_h^c}$$

induces a homeomorphism when passing to the quotient by $N_{\mathbb{C}}$, where on the left we consider the geometric quotient and on the right the quotient by the equivalence

relation (14). We can therefore identify the piece \mathcal{T}_F with the orbit space $(\mathbb{C}^*)^{F^c}/N_{\mathbb{C}}$. To conclude the proof of the theorem we need to show that the $(\mathbb{C}^*)^{F^c}/N_{\mathbb{C}}$ is a complex quasifold of dimension p , where p is the dimension of F . Choose an $I \in \mathcal{I}$ such that $\text{card}(I_F \cap I) = n - p$ and denote by

$$\check{\Gamma}_I = \Gamma_I / (\Gamma_I \cap T^F). \quad (16)$$

The action of $\check{\Gamma}_I$ on $(\mathbb{C}^*)^{I \setminus I \cap I_F}$ is well defined. Now consider the mapping $p_F : (\mathbb{C}^*)^{I \setminus I \cap I_F} \rightarrow (\mathbb{C}^*)^{F^c}$ defined by

$$(p_F(\underline{z}))_j = \begin{cases} z_j & \text{if } j \in I \setminus I \cap I_F \\ 1 & \text{if } j \in (I \cup I_F)^c \\ 0 & \text{if } j \in I_F \end{cases}$$

The mapping p_F induces a bijective continuous mapping

$$\phi_F : (\mathbb{C}^*)^{I \setminus I \cap I_F} / \check{\Gamma}_I \rightarrow (\mathbb{C}^*)^{F^c} / N_{\mathbb{C}}. \quad (17)$$

The mapping ϕ_F is also open since the mapping

$$\begin{aligned} \mathbb{C}^{(I \cup I_F)^c} \times (\mathbb{C}^*)^{I \setminus I \cap I_F} &\longrightarrow (\mathbb{C}^*)^{F^c} \\ (\underline{w}, \underline{z}) &\longmapsto (\exp(\underline{w}) \exp(\pi_I^{-1}(\pi(\underline{w}))) p_F(\underline{z})) \end{aligned}$$

is not only surjective but has surjective differential at every point. Therefore ϕ_F is a homeomorphism and $(\mathbb{C}^*)^{I \setminus I \cap I_F} / \check{\Gamma}_I$ gives a chart covering the p -dimensional complex quasifold \mathcal{T}_F . \square

Corollary 3.2 *The singular stratum \mathcal{T}_F corresponding to a singular face F can be identified with the orbit space $(\mathbb{C}^*)^{F^c}/N_{\mathbb{C}}$, which is precisely, by (6), the $D_{\mathbb{C}}$ -orbit corresponding to the face F . The maximal stratum is the union of the orbits of $D_{\mathbb{C}}$ corresponding to the regular faces, in particular, it contains the orbit corresponding to the interior of the polytope.*

Proof. See proof of Theorem 3.1.

Corollary 3.3 *The projection mapping $\mathbb{C}_{\Delta}^d \rightarrow \mathbb{C}_{\Delta}^d / N_{\mathbb{C}}$ is open.*

Proof. The statement can be easily proved by making use of (15).

Proposition 3.4 *The n -dimensional complex quasitorus $D_{\mathbb{C}}$ acts continuously on X , with a dense open orbit. Moreover the restriction of the $D_{\mathbb{C}}$ -action to each piece of the space X is holomorphic.*

Proof. The proof is a simple consequence of Corollary 3.2, see also the proof of the analogous result in [4, 1]. The dense open orbit is the one corresponding to the interior of the polytope.

3.2 Building blocks: links and complex cones

Consider the singular p -dimensional piece \mathcal{T}_F . We show that all of the points of \mathcal{T}_F have the same link. We first describe this link on the polytope Δ . Let $j_F : \mathfrak{d}_F \hookrightarrow \mathfrak{d}$ be the inclusion mapping. Define

$$\Sigma_F^\diamond = \bigcap_{j \in I_F} \{ \mu \in \mathfrak{d}^* \mid \langle \mu, X_j \rangle \geq \lambda_j \}.$$

By projecting Σ_F^\diamond onto \mathfrak{d}_F^* we obtain the cone

$$\Sigma_F = j_F^*(\Sigma_F^\diamond), \quad (18)$$

it is an $(n - p)$ -dimensional cone in \mathfrak{d}_F^* with vertex $j_F^*(F)$. If G is a q -dimensional face of Δ containing F , then $j_F^*(G)$ is a $(q - p)$ -dimensional face of Σ_F . By slicing the cone Σ_F with a hyperplane transversal to its faces we obtain a polytope Δ_F which is the link of the face F . Each face of Δ containing F gives rise to a face of Δ_F . More precisely: for each $j \in I_F$ we can find an $s_j \in (0, 1]$ such that, taken $X_0 = \sum_{j \in I_F} s_j X_j$, the intersection

$$\Delta_F = \Sigma_F \cap \{ \xi \in \mathfrak{d}_F^* \mid \langle \xi, X_0 \rangle = \sum_{j \in I_F} \lambda_j s_j + 1 \}$$

is a nonempty convex polytope of dimension $(n - p - 1)$. Let G be a q -dimensional face of Δ properly containing F , then

$$j_F^*(G) \cap \{ \xi \in \mathfrak{d}_F^* \mid \langle \xi, X_0 \rangle = \sum_{j \in I_F} \lambda_j s_j + 1 \}$$

is a $(q - p - 1)$ -dimensional face of Δ_F , which is singular in Δ_F if and only if G is singular in Δ . The complex spaces corresponding to these newly defined convex polyhedra are the building blocks of our stratified space. For each point in \mathcal{T}_F the cone $C(L)$ of Definition 1.5 will be the complex space corresponding to the polyhedral cone Σ_F , whilst the space Y of Definition 1.7 will be the complex space corresponding to Δ_F . Let us now construct these spaces. Recall that $\mathfrak{d}_F = \text{Span}\{X_j \mid j \in F\}$. Notice first that $Q_F = \mathfrak{d}_F \cap Q$ is a quasilattice. Consider the convex polyhedral cone Σ_F , together with the normals X_j , with $j \in I_F$, and the quasilattice Q_F . We have the short exact sequence

$$0 \longrightarrow \mathfrak{n}^F \xrightarrow{\iota_F} \mathbb{R}^F \xrightarrow{\pi_F} \mathfrak{d}_F \longrightarrow 0 \quad (19)$$

then

$$N^F = \text{Ker}(T^F \longrightarrow \mathfrak{d}_F/Q_F).$$

Remark that $N^F = N \cap T^F$ with $\dim N^F = r_F - n + p$. With the procedure described in Section 3 we construct the quotient

$$C(L_F) = \mathbb{C}_{\Sigma_F}^F // N_{\mathbb{C}}^F = \mathbb{C}^F // N_{\mathbb{C}}^F$$

this is a space decomposed by complex quasifolds, the decomposition is the one induced by that of Σ_F in regular and singular faces. Remark that Σ_F is not a polytope, it is

a polyhedral cone, but the construction described in the previous sections applies with no changes; notice also $\mathbb{C}_{\Sigma_F}^F = \mathbb{C}^F$.

Now let $\text{ann}(X_0)$ be the annihilator of X_0 and let

$$\text{ann}(X_0) \xrightarrow{k_F} (\mathfrak{d}_F)^*$$

be the natural inclusion. Then k_F^* projects \mathfrak{d}_F onto the $(n - p - 1)$ -dimensional space $(\text{ann}(X_0))^* \simeq \mathfrak{d}_F / \langle X_0 \rangle$. We continue to denote by Δ_F the polytope Δ_F viewed in the subspace $\text{ann}(X_0)$:

$$\Delta_F = \bigcap_{j \in I_F} \{ \xi \in \text{ann}(X_0) \mid \langle \xi, k_F^*(X_j) \rangle \geq \lambda_j - \langle \xi_0, X_j \rangle \} \quad (20)$$

where ξ_0 is a point in the affine hyperplane with which we cut Σ_F . Now apply the construction described in Section 3 to the polytope Δ_F , with the choice of normals $k_F^*(X_j)$ and quasilattice $Q_{F,0} = k_F^*(\mathfrak{d}_F \cap Q)$. We obtain the exact sequence

$$0 \longrightarrow \mathfrak{n}_0^F \xrightarrow{\iota_{F,0}} \mathbb{R}^F \xrightarrow{k_F^* \circ \pi_F} (\text{ann}(X_0))^* \longrightarrow 0 \quad (21)$$

Let $\mathfrak{s} = \text{Span}\{s'_1, \dots, s'_d\}$ where $s'_j = s_j$ if $j \in I_F$ and $s'_j = 0$ otherwise. It is easy to check that

$$\mathfrak{n}_0^F = \mathfrak{n}^F \oplus \mathfrak{s}.$$

The subgroup of T^F that we need is then

$$N_0^F = \text{Ker} \left(T^F \longrightarrow (\text{ann}(X_0))^* / Q_{F,0} \right).$$

Remark that

$$N_0^F / N^F \simeq \exp(\mathfrak{s}).$$

Notice that the polytope Δ_F , obtained by slicing the cone Σ_F , is combinatorially equivalent to $\Sigma_F \setminus \{\text{cone point}\}$, therefore $\mathbb{C}_{\Delta_F}^F$ does not depend on the choice of X_0 , whilst the group N_0^F does. The space corresponding to Δ_F with this set of data is the space X_{Δ_F} , decomposed by complex quasifolds, given by the quotient

$$\mathbb{C}_{\Delta_F}^F / (N_0^F)_{\mathbb{C}}$$

Lemma 3.5 *The natural mapping*

$$s_F : \mathbb{C}_{\Sigma_F}^F / N_{\mathbb{C}}^F \setminus \{[0]\} \longrightarrow \mathbb{C}_{\Delta_F}^F / (N_0^F)_{\mathbb{C}}$$

satisfy Definition 1.7.

Proof. We have already observed that $(N_0^F)_{\mathbb{C}} / N_{\mathbb{C}}^F \simeq \exp(\mathfrak{s}_{\mathbb{C}})$, therefore the continuous mapping s_F is surjective, with fibre isomorphic to the quotient of the complex group $\exp(\mathfrak{s}_{\mathbb{C}})$ by the stabilizer of $\exp(\mathfrak{s}_{\mathbb{C}})$ on the fiber itself. Moreover s naturally respects the decomposition and is holomorphic when restricted to each piece. \square

Remark 3.6 The coefficients s_j can always be chosen in such a way that the group $\exp(\mathfrak{s})$ is compact or acts freely on \mathbb{C}^F , nonetheless, since from the examples it is clear that there are choices of \mathfrak{s} that are natural (see [1, Example 3.6]), we prefer to have freedom of choice on the coefficients and to leave Definition 1.7 as it is, with no stronger requirements on the action of S .

3.3 Holomorphic local triviality

Let us first define and analyse closely the candidate product space. The action of the finitely generated group $\check{\Gamma}_I$ on the product $(\mathbb{C}^*)^{I \setminus (I \cap I_F)} \times (\mathbb{C}^F // N_{\mathbb{C}}^F)$ is defined by $[\gamma](\underline{z}, [\underline{w}]) = (\gamma \underline{z}, [\gamma \underline{w}])$. Observe that the action is well defined and it is free on the first factor. Now define on the product $(\mathbb{C}^*)^{I \setminus (I \cap I_F)} \times \mathbb{C}^F$ the following equivalence relation:

$$(\underline{w}_1, \underline{z}_1) \sim_1 (\underline{w}_2, \underline{z}_2) \quad (22)$$

if and only if there exists $\gamma \in \Gamma_I$ such that

$$\gamma \underline{w}_1 = \underline{w}_2$$

and

$$\gamma \underline{z}_1 \sim_{N^F} \underline{z}_2.$$

It is easy to check that

$$\left((\mathbb{C}^*)^{I \setminus (I \cap I_F)} \times \mathbb{C}^F // N_{\mathbb{C}}^F \right) / \check{\Gamma}_I \simeq (\mathbb{C}^*)^{I \setminus (I \cap I_F)} \times \mathbb{C}^F / \sim_1 \quad (23)$$

Consider now the action of the quotient group $N_{\mathbb{C}} / N_{\mathbb{C}}^F$ on the product $(\mathbb{C}^*)^{F^c} \times (\mathbb{C}^F // N_{\mathbb{C}}^F)$ defined by

$$[g] \cdot (\underline{w}, [\underline{z}]) = (g \underline{w}, [g \underline{z}])$$

where $g \in N_{\mathbb{C}}$, $\underline{w} \in (\mathbb{C}^*)^{F^c}$ and $\underline{z} \in \mathbb{C}^F$. On $(\mathbb{C}^*)^{F^c} \times \mathbb{C}^F$ define the following equivalence relation:

$$(\underline{w}_1, \underline{z}_1) \sim_2 (\underline{w}_2, \underline{z}_2) \quad (24)$$

if and only if there exists $g \in N_{\mathbb{C}}$ such that

$$g \underline{w}_1 = \underline{w}_2$$

and

$$g \underline{z}_1 \sim_{N^F} \underline{z}_2.$$

It is easy to check that

$$\left((\mathbb{C}^*)^{F^c} \times \mathbb{C}^F // N_{\mathbb{C}}^F \right) / (N_{\mathbb{C}} / N_{\mathbb{C}}^F) \simeq \left((\mathbb{C}^*)^{F^c} \times \mathbb{C}^F \right) / \sim_2 \quad (25)$$

We want to prove the following

Lemma 3.7 *The spaces*

$$\left((\mathbb{C}^*)^{I \setminus (I \cap I_F)} \times \mathbb{C}^F \right) / \sim_1$$

and

$$\left((\mathbb{C}^*)^{F^c} \times \mathbb{C}^F \right) / \sim_2$$

are diffeomorphic as decomposed spaces, moreover the diffeomorphism is a biholomorphism when restricted to the pieces.

Proof. First of all remark that, since the projection $\mathbb{C}^F \rightarrow \mathbb{C}^F / N_{\mathbb{C}}^F$ is open by Corollary 3.3, we have that the projections

$$(\mathbb{C}^*)^{I \setminus (I \cap I_F)} \times \mathbb{C}^F \rightarrow ((\mathbb{C}^*)^{I \setminus (I \cap I_F)} \times \mathbb{C}^F) / \sim_1$$

and

$$(\mathbb{C}^*)^{F^c} \times \mathbb{C}^F \rightarrow ((\mathbb{C}^*)^{F^c} \times \mathbb{C}^F) / \sim_2$$

are both open. The candidate diffeomorphism between the spaces in the statement is the mapping f defined by $f([\underline{w}, [\underline{z}]] = [\underline{w} + \underline{1}, [\underline{z}]]$ where $\underline{w} \in \mathbb{C}^{I \setminus I \cap I_F}$, $\underline{z} \in \mathbb{C}^F$ and $\underline{1} \in \mathbb{C}^{(I \cap I_F)^c}$ is defined by

$$\underline{1}_j = 1 \quad \text{for } j \notin I \cup I_F \quad \underline{1}_j = 0 \quad \text{for } j \in I \cup I_F. \quad (26)$$

It is easy to check that f is bijective, in particular injectivity is due to the fact that

$$N_{\mathbb{C}} \cap T^{I \cup I_F} = \Gamma_I N_{\mathbb{C}}^F.$$

It is also a straightforward check to prove that f is continuous. The key point is to prove that f is open. Let U be an open subset of $(\mathbb{C}^*)^{F^c} \times \mathbb{C}^F$, saturated with respect to \sim_2 and let $W \times \underline{1} \times V$ be contained in U , with W an open subset of $(\mathbb{C}^*)^{I \setminus I \cap I_F}$ and V an open subset of \mathbb{C}^F , invariant under the actions of N^F and Γ_I . In order to prove that f is open it suffices to prove that

$$N_{\mathbb{C}}(W \times \underline{1} \times V)$$

is open (it is obviously contained in U). Consider the mapping

$$\begin{aligned} W \times V \times \mathbb{C}^{(I \cap I_F)^c} &\longrightarrow (\mathbb{C}^*)^{F^c} \times \mathbb{C}^F \\ ((\underline{w}, \underline{z}), V) &\longmapsto \exp(V) \exp(\pi_I^{-1}(\pi(V)))(\underline{w}, \underline{1}, \underline{z}) \end{aligned} \quad (27)$$

the image of this mapping is a subset of $N_{\mathbb{C}}(W \times \underline{1} \times V)$, since by Lemma 2.4

$$\exp(V) \exp(\pi_I^{-1}(\pi(V))) \in N_{\mathbb{C}}.$$

We prove that it is exactly $N_{\mathbb{C}}(W \times \underline{1} \times V)$: take an element $g \in N_{\mathbb{C}}$, then, by Lemma 2.4, there exist $\gamma \in \Gamma_I$ and $Z \in \mathbb{C}^{I^c}$ such that $g = \gamma \exp(Z) \exp(\pi_I^{-1}(\pi(Z)))$. Split $Z = Z_1 + Z_2$ according to the decomposition $\mathbb{C}^{I^c} = \mathbb{C}^{(I \cup I_F)^c} \times \mathbb{C}^{I_F \setminus (I_F \cap I)}$, then

$$g = \gamma \exp(Z_1) \exp(\pi_I^{-1}(\pi(Z_1))) \exp(Z_2) \exp(\pi_I^{-1}(\pi(Z_2))).$$

Now observe that

$$\exp(Z_2) \exp(\pi_I^{-1}(\pi(Z_2))) = \exp(Z_2) \exp(\pi_{I \cap I_F}^{-1}(\pi_F(Z_2)))$$

is an element of $N_{\mathbb{C}}^F$ and recall that the open subset V is invariant by the action of $N_{\mathbb{C}}^F$ and by the action of Γ_I . Therefore $g(\underline{w}, \underline{1}, \underline{z}) = \exp(Z_1) \exp(\pi_I^{-1}(\pi(Z_1)))(\underline{w}', \underline{1}, \underline{z}')$ with $(\underline{w}', \underline{z}') \in W \times V$. This proves our assertion that the saturated of $W \times \underline{1} \times V$ under the action of $N_{\mathbb{C}}$ is exactly the image of the mapping (27), on the other hand it is easy to check that the differential of the mapping (27) is never zero, its image is therefore open. It is now straightforward to check that the restriction of f to the pieces is a biholomorphism. \square

We are now ready to prove the local holomorphic triviality of our decomposition:

Lemma 3.8 (Holomorphic local triviality) *Recall that $\mathcal{T}_F \simeq (\mathbb{C}^*)^{I \setminus I \cap I_F} / \check{\Gamma}_I$ and that $((\mathbb{C}^*)^{F^c} \times \mathbb{C}^F) // N_{\mathbb{C}}$ is an open subset of X_{Δ} containing \mathcal{T}_F . There exists a mapping h_F from $((\mathbb{C}^*)^{I \setminus (I \cap I_F)} \times \mathbb{C}^F // N_{\mathbb{C}}^F) / \check{\Gamma}_I$ onto $((\mathbb{C}^*)^{F^c} \times \mathbb{C}^F) // N_{\mathbb{C}}$, which is a homeomorphism and a biholomorphism restricted to the pieces of the respective decompositions.*

Proof. By (23, 25) and Lemma 3.7 it suffices to prove that the mapping

$$h_F : (\mathbb{C}^*)^{F^c} \times \mathbb{C}^F / \sim_2 \longrightarrow ((\mathbb{C}^*)^{F^c} \times \mathbb{C}^F) // N_{\mathbb{C}}$$

is a homeomorphism and a biholomorphism restricted to the pieces of the respective decompositions. Consider two points $(\underline{w}, \underline{z})$ and $(\underline{w}', \underline{z}')$ in $(\mathbb{C}^*)^{F^c} \times \mathbb{C}^F$. Then $(\underline{w}', \underline{z}') \sim_N (\underline{w}, \underline{z})$ if and only if there exists an element $g \in N_{\mathbb{C}}$ and a face G in Δ such that $(g(\underline{w}' + \underline{z}'))_{G^c} = (\underline{w}, \underline{z})_{G^c}$, if and only if $g\underline{w}' = \underline{w}$ and $(g\underline{z}')_{G^c} = \underline{z}_{G^c}$ if and only if $g\underline{w}' = \underline{w}$ and $g\underline{z}' \sim_{N^F} \underline{z}$ if and only if $(\underline{w}', \underline{z}') \sim_2 (\underline{w}, \underline{z})$. It is then easy to check that the mapping h_F is a homeomorphism and a biholomorphism when restricted to pieces. \square

3.4 The interplay with the symplectic setup

In this section we will prove the statement that was used in the proof of Theorem 2.6. Moreover, in order to conclude that $\mathbb{C}_{\Delta}^d // N_{\mathbb{C}}$ is a complex stratified space, we need to prove that the spaces $\mathbb{C}^F // N_{\mathbb{C}}^F$ are cones over a compact stratified space L satisfying the definition of link, moreover all of the mappings involved have to satisfy the requirements of the Definitions 1.5, 1.7. General results, for example on bundles, are not of immediate application to our spaces because of their topology; therefore, although a description of the building blocks of our stratification can be given within our set up (see the first row of diagram 37), we make use of the interplay with the symplectic quotients in order to give a neat description of the link L_F and of X_{Σ_F} as a real cone over it. Hence, before going on to describe the cone X_{Σ_F} , we will briefly recall from [1] the symplectic construction for the nonsimple case (cf. [18] for the simple case).

Consider the convex sets Δ , Σ_F and Δ_F . We have already defined the groups N , N^F and $N_0^F = N^F \exp(\mathfrak{s})$. We briefly recall the construction of the moment mappings relative to these polyhedral sets. A moment mapping with respect to the standard action of the torus T^d on \mathbb{C}^d is given by

$$\Upsilon(\underline{z}) = \sum_{j=1}^d (|z_j|^2 + \lambda_j) e_j^*, \quad (28)$$

where the λ_j 's are given in (1). Analogously we consider the mappings on \mathbb{C}^F given by

$$\Upsilon_F(\underline{z}) = \sum_{j \in I_F} (|z_j|^2 + \lambda_j) e_j^*, \quad (29)$$

and

$$\Upsilon_{F,0}(\underline{z}) = \sum_{j \in I_F} (|z_j|^2 + \lambda'_j) e_j^*, \quad (30)$$

with $\lambda'_j = \lambda_j - \langle \xi_0, X_j \rangle$.

A moment mapping with respect to the induced Hamiltonian action of N on \mathbb{C}^d is then: $\Psi_\Delta : \mathbb{C}^d \rightarrow \mathfrak{n}^*$ given by

$$\Psi_\Delta = \iota^* \circ \Upsilon.$$

In the same manner the mappings

$$\Psi_{\Sigma_F} = \iota_F^* \circ \Upsilon_F$$

and

$$\Psi_{\Delta_F} = \iota_{F,0}^* \circ \Upsilon_{F,0}$$

are moment mappings relative to the Hamiltonian action of the $(r_F - n + p)$ -dimensional group N^F on \mathbb{C}^F and to the Hamiltonian action of the $(r_F - n + p + 1)$ -dimensional group N_0^F on \mathbb{C}^F respectively. The explicit expression of these moment mappings are

$$\begin{aligned} \Psi_\Delta : \mathbb{C}^d &\longrightarrow \mathfrak{n}^* \\ \underline{z} &\longmapsto \sum_{j=1}^d (|z_j|^2 + \lambda_j) \iota^*(e_j^*) \\ \Psi_{\Sigma_F} : \mathbb{C}^F &\longrightarrow \mathfrak{n}_F^* \\ \underline{z} &\longmapsto \sum_{j \in I_F} |z_j|^2 \iota_F^*(e_j^*) \end{aligned} \quad (31)$$

and finally

$$\begin{aligned} \Psi_{\Delta_F} : \mathbb{C}^F &\longrightarrow \mathfrak{n}_F^* \oplus \mathbb{R} \\ \underline{z} &\longmapsto (\sum_{j \in I_F} |z_j|^2 \iota_F^*(e_j^*), \sum_{j \in I_F} s_j |z_j|^2 - 1) \end{aligned} \quad (32)$$

Where in (32) we have identified $(\mathfrak{n} \oplus \mathfrak{s})^* \simeq \mathfrak{n}^* \oplus \mathbb{R}$. The symplectic quotient corresponding to Δ is then the reduced space

$$M_\Delta = \Psi_\Delta^{-1}(0)/N,$$

whilst for Σ_F and Δ_F we have

$$M_{\Sigma_F} = \Psi_{\Sigma_F}^{-1}(0)/N^F$$

and

$$M_{\Delta_F} = \Psi_{\Delta_F}^{-1}(0)/N_0^F$$

respectively. These symplectic quotients are spaces stratified by symplectic quasifolds, they are endowed with the continuous effective Hamiltonian action of the quasitori \mathfrak{d}/Q , \mathfrak{d}_F/Q_F and $(\text{ann}(X_0))^*/Q_{F,0}$ respectively. The quasitori act smoothly on the strata, with moment mappings given by the restriction to the strata of the mappings $\Phi_\Delta = (\pi^*)^{-1} \circ \Upsilon$, $\Phi_{\Sigma_F} = (\pi_F^*)^{-1} \circ \Upsilon_F$ and $\Phi_{\Delta_F} = (\pi_F^* \circ k_F)^{-1} \circ \Upsilon_{F,0}$ respectively. The constants in (28,29,30) have been chosen in such a way that the images of the mappings Φ_Δ , Φ_{Σ_F} and Φ_{Δ_F} are respectively Δ , Σ_F and Δ_F [1].

In order to relate the complex and symplectic quotients we need to prove the following statement, which is an adaptation to our case of the results contained in [12, Appendix 1], to which we refer for further details.

Lemma 3.9 *The mapping $\Psi_\Delta : \mathbb{C}^d \rightarrow \mathfrak{n}^*$ satisfies the following properties:*

1. $\Psi_{\Delta}^{-1}(0) \subset \cup_F (\mathbb{C}^*)^{F^c}$; moreover, for each face F , the set $(\mathbb{C}^*)^{F^c}$ intersects $\Psi_{\Delta}^{-1}(0)$ in at least one point;
2. let \underline{z} be any point in \mathbb{C}_{Δ}^d , then $\Psi(A\underline{z})$ is a cone, which is open as a subset of a suitable linear subspace of \mathfrak{n}^* ; the cone depends only on the set $I_{\underline{z}} = \{j \mid z_j = 0\}$;
3. let $\underline{z} \in \mathbb{C}_{\Delta}^d$, then there exists an $a \in A$ such that $\Psi_{\Delta}(a\underline{z}) = 0$ if and only if $\underline{z} \in (\mathbb{C}^*)^{F^c}$ for some face F of Δ ;
4. Ψ_{Δ} is a proper mapping, in particular $\Psi_{\Delta}^{-1}(0)$ is compact.

Proof. Let $\underline{z} \in \mathbb{C}^d$. We have that $\Psi(\underline{z}) = 0$ if and only if $\iota^*(\sum_{j=1}^d (|z_j|^2 + \lambda_j)e_j^*) = 0$ if and only if, by (10), there exists a (unique) point $\xi \in \mathfrak{d}^*$ such that

$$\pi^*(\xi) = \sum_{j=1}^d (|z_j|^2 + \lambda_j)e_j^*$$

if and only if for every $k = 1, \dots, d$ we have

$$\langle \xi, X_k \rangle = |z_k|^2 + \lambda_k,$$

if and only if $\xi \in \Delta$. This implies point 1.

Denote by α_k the elements of \mathfrak{n}^* given by $\alpha_k = -2\pi\iota^*(e_k^*)$. Consider the stabilizer of A at \underline{z} , given by $\exp(i(\mathfrak{n} \cap \mathbb{R}^{I_{\underline{z}}}))$ and let \mathfrak{r} be the orthogonal complement of $\mathfrak{n} \cap \mathbb{R}^{I_{\underline{z}}}$ in \mathfrak{n} . Finally consider the subset of \mathfrak{n}^* given by $\text{Span}\{\alpha_k \mid k \notin I_{\underline{z}}\}$. Now remark that

$$\text{Span}\{\alpha_k \mid k \notin I_{\underline{z}}\} \simeq \text{ann}(\mathfrak{n} \cap \mathbb{R}^{I_{\underline{z}}}) \simeq \mathfrak{r}^*$$

where $\text{ann}(\mathfrak{n} \cap \mathbb{R}^{I_{\underline{z}}})$ denotes the annihilator of $\mathfrak{n} \cap \mathbb{R}^{I_{\underline{z}}}$. The subset $\Psi(A\underline{z})$ of \mathfrak{n}^* can be identified with the image of the mapping

$$\begin{array}{ccc} \mathfrak{r} & \longrightarrow & \mathfrak{r}^* \\ Y & \longmapsto & \sum_{k \notin I_{\underline{z}}} e^{\alpha_k(Y)} |z_k|^2 \alpha_k + \lambda \end{array} \quad (33)$$

where $\lambda = \sum_{k=1}^d \lambda_k \iota^*(e_k^*)$. Remark now that the mapping above is the Legendre transform of the function

$$\begin{array}{ccc} F_{\underline{z}} : \mathfrak{r} & \longrightarrow & \mathbb{R} \\ Y & \longmapsto & \sum_{k \notin I_{\underline{z}}} e^{\alpha_k(Y)} |z_k|^2 + \lambda(Y) \end{array} .$$

It is easy to check that the Hessian of $F_{\underline{z}}$ is positive definite, since the α_k 's generate \mathfrak{r}^* , therefore $F_{\underline{z}}$ is strictly convex. This implies that the image of the mapping (33), namely $\Psi(A\underline{z})$, is the open convex cone in $\text{Span}\{\alpha_k \mid k \notin I_{\underline{z}}\}$ given by

$$\left\{ \sum_{k \notin I_{\underline{z}}} t_k \alpha_k + \lambda \mid t_k > 0 \right\}$$

moreover \mathfrak{r} is mapped diffeomorphically onto the open cone by the mapping (33). This proves point 2.

Now we are able to prove point 3: let $\underline{z} \in (\mathbb{C}^*)^{F^c}$ for some F in Δ . By point (i) there exists a point \underline{w} in $(\mathbb{C}^*)^{F^c}$ such that the A -orbit $A\underline{w}$ intersects the zero set $\Psi^{-1}(0)$, therefore the cone in \mathfrak{n}^* corresponding to the orbit \underline{w} must contain 0. But by point (ii) this holds also for the orbit $A\underline{z}$. By the same kind of argument we can deduce that, if \underline{z} is not in $(\mathbb{C}^*)^{F^c}$ for some face F , then the A -orbit $A\underline{z}$ does not intersect the zero set $\Psi^{-1}(0)$.

In order to prove the last point of our statement, let us first prove that the zero set is compact. As we have already remarked $\Psi(\underline{z}) = 0$ if and only if $\Upsilon(\underline{z}) \in \text{Ker}(\iota^*)$. Therefore $\Psi^{-1}(0) = \Upsilon^{-1}(\text{Im}(\Upsilon) \cap \text{Ker} \iota^*)$. But $\text{Im}(\Upsilon) \cap \text{Ker} \iota^* = \{\xi \in (\mathbb{R}^d)^* \mid \langle \xi, e_k \rangle \geq \lambda_k, \ k = 1 \dots, d\} \cap \text{Im}(\pi^*)$. This can be deduced from the definition of Υ together with (10). Therefore $\text{Im}(\Upsilon) \cap \text{Ker}(\iota^*)$ is exactly $\pi^*(\Delta)$, in particular it is compact. This implies, since Υ is proper, that $\Psi^{-1}(0)$ is compact.

Observe now that, since ι^* is a linear projection and $\text{Im}(\Upsilon)$ is the positive orthant shifted by λ , we have that $\text{Im}(\Upsilon) \cap (\iota^*)^{-1}(\eta)$ is compact for every $\eta \in \mathfrak{n}^*$. And again properness of Υ implies that Ψ is also proper. \square

Corollary 3.10 *Let F be a face of Δ , let \mathfrak{r} be the orthogonal complement of $\mathfrak{n} \cap \mathbb{R}^F$ in \mathfrak{n} and let \underline{z} be a point in $(\mathbb{C}^*)^{F^c}$. Then there exists a unique point $\underline{x} \in \Psi^{-1}(0)$, a unique point $\xi \in F$ and a unique $Y \in \mathfrak{r}$ such that*

$$A\underline{z} \cap \Psi^{-1}(0) = \{\underline{x}\},$$

$$\exp(iY)\underline{z} = \underline{x}$$

and

$$\langle \xi, X_j \rangle - \lambda_j = |x_j|^2, \quad j = 1, \dots, d$$

Proof. The argument is the same used in [4, Remark 3.3]. Suppose we have two points of intersection: \underline{x}_1 and \underline{x}_2 , then there is an $a \in A$ such that $a\underline{x}_1 = \underline{x}_2$, but either a is in the stabilizer of \underline{x}_1 , and therefore $\underline{x}_1 = \underline{x}_2$, or a is not in the stabilizer of \underline{x}_1 . In this case it moves \underline{x}_1 out of $\Psi^{-1}(0)$, contradiction. \square

Corollary 3.11 *Let \underline{z} be a point in \mathbb{C}_Δ^d , then there exists a unique point $\underline{x} \in \Psi^{-1}(0)$ such that $\overline{A\underline{z}} \cap \Psi^{-1}(0) = \{\underline{x}\}$.*

Proof. The statement follows by Corollary 3.10 and by Theorem 2.6 \square The inclusion mapping $\Psi_\Delta^{-1}(0) \hookrightarrow \mathbb{C}_\Delta^d$ induces a mapping

$$\chi_\Delta : \Psi_\Delta^{-1}(0)/N \longrightarrow \mathbb{C}_\Delta^d/N_{\mathbb{C}} \quad (34)$$

between the two quotients. By Corollary 3.11 we can also define the surjective mapping

$$\begin{aligned} \Xi_\Delta & : \mathbb{C}_\Delta^d & \longrightarrow & \Psi^{-1}(0) \\ & \underline{z} & \longrightarrow & \overline{A\underline{z}} \cap \Psi^{-1}(0). \end{aligned} \quad (35)$$

This implies that χ_Δ is bijective and $\chi_\Delta^{-1}([\underline{z}]) = [\Xi_\Delta(\underline{z})]$.

Lemma 3.12 *Suppose that the mapping χ_Δ is a homeomorphism and take a converging sequence $\underline{z}_n \longrightarrow \underline{z}$ in \mathbb{C}_Δ^d , then the sequence $\Xi_\Delta(\underline{z}_n)$ converges to $\Xi_\Delta(\underline{z})$.*

The proof is direct and is left to the reader.

Lemma 3.13 *Let \mathcal{T}_F be a singular stratum, corresponding to the singular face F . The complex and symplectic quotients that express the singular stratum in the symplectic and complex setup are naturally isomorphic.*

Proof. Let I be such that $\text{card}(I \cap I_F) = n - p$ and let a_{ij} be the $d \times n$ matrix of the projection $\pi : \mathbb{R}^d \rightarrow \mathfrak{d}$ with respect to the standard basis and the basis $\{X_j \mid j \in I\}$. Then recall that the stratum \mathcal{T}_F , given by the quotient $(\Psi^{-1}(0) \cap (\mathbb{C}^*)^{F^c})/N$, can be identified with the quotient $\tilde{B}_F/\check{\Gamma}_{I \setminus (I \cap I_F)}$, where

$$\tilde{B}_F = \{\underline{z} \in (\mathbb{C}^*)^{I \setminus (I \cap I_F)} \mid \sum_{h \in I} a_{hk}(|z_h|^2 + \lambda_h) - \lambda_k > 0, k \in (I \cup I_F)^c\}.$$

We have to prove that the mapping $(\chi_F)_{loc}$ in the following diagram is a diffeomorphism:

$$\begin{array}{ccc} (\mathbb{C}^*)^{I \setminus (I \cap I_F)} / \check{\Gamma}_{I \setminus (I \cap I_F)} & \xrightarrow{\phi_F} & (\mathbb{C}^*)^{F^c} / N_{\mathbb{C}} \\ (\chi_F)_{loc} \uparrow & & \uparrow \chi_F \\ B / \check{\Gamma}_{I \setminus (I \cap I_F)} & \xrightarrow{\phi_F^s} & \Psi^{-1}(0) \cap (\mathbb{C}^*)^{F^c} / N \end{array} \quad (36)$$

The mapping χ_F is induced by the inclusion of $\Psi^{-1}(0) \cap (\mathbb{C}^*)^{F^c}$ into $(\mathbb{C}^*)^{F^c}$; the homeomorphism ϕ_F was defined in (17); the homeomorphism ϕ_F^s is defined by $\phi_F^s([x]) = [x + \tilde{x}]$, where $\tilde{x} \in \mathbb{C}^{(I \cup I_F)^c}$ is given by

$$\tilde{x}_k = \sqrt{\sum_{h \in I} a_{hk}(|x_h|^2 + \lambda_h) - \lambda_k}$$

[1]; The mapping $(\chi_F)_{loc}$ is the diffeomorphism that sends $\underline{z} \in B$ to

$$-\exp(\pi_I^{-1}(\pi(X)))_{I \setminus (I \cap I_F)} \underline{x}$$

where

$$X_k = -\frac{1}{2\pi} \log\left(\sum_{h \in I} a_{hk}(|x_h|^2 + \lambda_h) - \lambda_k\right)$$

for $k \in (I \cup I_F)^c$ and $X_k = 0$ otherwise. \square

We prove now for the symplectic case an analogous of Lemma 3.7:

Lemma 3.14 *The quotients*

$$(\tilde{B}_F \times \Psi^{-1}(0)/N^F)/\check{\Gamma}_I$$

and

$$\left((\Psi^{-1}(0) \cap (\mathbb{C}^*)^{F^c}) \times \Psi_{\Sigma_F}^{-1}(0)/N^F\right)/(N/N_F)$$

are diffeomorphic.

Proof. The action of N/N^F is defined as follows: consider $[g] \in N/N^F$ and $(\underline{x}, [\underline{z}]) \in (\Psi^{-1}(0) \cap (\mathbb{C}^*)^{F^c}) \times \Psi_{\Sigma_F}^{-1}(0)/N^F$, then $[g](\underline{x}, [\underline{z}]) = (g\underline{x}, [g\underline{z}])$. It is easy to check that this action is well defined. Consider now the mapping:

$$\begin{aligned} (\tilde{B}_F \times \Psi^{-1}(0)/N^F)/\Gamma_{I \setminus (I \cap I_F)} &\longrightarrow \Psi^{-1}(0) \cap (\mathbb{C}^*)^{F^c} \times \Psi_{\Sigma_F}^{-1}(0)/N^F/(N/N_F) \\ [\underline{x}, [\underline{z}]] &\longmapsto [\underline{x} + \phi_F(\underline{x}), [\underline{z}]] \end{aligned}$$

It is straightforward to check that this mapping is bijective and continuous. The argument for the proof that it is closed is also straightforward, it is a direct consequence of Corollary 2.5. \square

3.5 The main results

We will complete in this section the proof of the two theorems below, they will be proved together as proofs are entangled. We shall proceed by induction on the depth of the polytope Δ , combining the complex and symplectic setup. As final result we obtain not only the proof that our decomposed space is indeed a stratification but also the prove that X_Δ is naturally isomorphic to its symplectic counterpart M_Δ .

Theorem 3.15 (X_Δ is a stratified space) *Let \mathfrak{d} be a real vector space of dimension n , and let $\Delta \subset \mathfrak{d}^*$ be a convex polytope. Choose inward-pointing normals to the facets of Δ , $X_1, \dots, X_d \in \mathfrak{d}$, and let Q be a quasilattice containing them. The quotient $X = \mathbb{C}_\Delta^d / N_\mathbb{C}$ is a complex stratified space, in particular for each singular face F of the polytope Δ there exist a compact link L_F and a complex link Y_F satisfying the requirements of the definitions 1.5, 1.7.*

Theorem 3.16 (The complex and symplectic quotients are isomorphic) *Let \mathfrak{d} be a vector space of dimension n , and let $\Delta \subset \mathfrak{d}^*$ be a convex polytope. Choose inward-pointing normals to the facets of Δ , $X_1, \dots, X_d \in \mathfrak{d}$, and let Q be a quasilattice containing them. Then the mapping*

$$\chi_\Delta : \Psi_\Delta^{-1}(0)/N \longrightarrow \mathbb{C}_\Delta^d / N_\mathbb{C}$$

is an equivariant homeomorphism with respect to the actions of D and $D_\mathbb{C}$ respectively. The restriction of χ_Δ to each stratum is a diffeomorphism of quasifolds. Moreover the induced symplectic form on each stratum is compatible with its complex structure, so that strata have the structure of Kähler quasifolds.

Proof. of Theorems 3.15, 3.16 We start by giving:

- a description of the real link L_F ;
- a characterization of the quotient $\mathbb{C}^F / (N_\mathbb{C}^F)$ as a real cone over L_F ;
- a diffeomorphism between complex and symplectic quotients at the cone level.

We first define the notion of depth of Δ . Let F be a singular face of the polytope Δ . We call singularity depth of F the minimum integer m such that there exists a sequence of faces $F_0 < \dots < F_j < \dots < F_m$ such that $F_0 = F$, F_m is regular and F_j is singular

for all $0 \leq j < m$. By definition the singularity depth of regular faces is set to be 0. We define the singularity depth of a polytope to be the maximum singularity depth attained by its faces. We then procede by induction on the polytope depth. Suppose that Δ has depth 0, namely Δ is a simple polytope. In this case, treated in [4], the complex quotient is a geometric quotient and the mapping $\chi_\Delta : \Psi^{-1}(0)/N \rightarrow \mathbb{C}_\Delta^d/N_\mathbb{C}$ is proved to be an equivariant diffeomorphism such that the induced symplectic structure on $\mathbb{C}_\Delta^d/N_\mathbb{C}$ is Kähler. Continuity of χ_Δ is straightforward; bijectivity of χ_Δ follows from Lemma 3.9. The mapping χ_Δ is then computed explicitly on the local charts and turns out to be a local diffeomorphism, which implies that its inverse is also continuous.

Suppose now that Theorems 3.15, 3.16 hold for polytopes of depth less than or equal to $n-1$, we want to prove that they hold for polytopes of depth n . Consider a polytope Δ of depth n . Let F be a p -dimensional singular face and let Σ_F and Δ_F be the convex sets associated to F as described in (18) and (20). Notice that, in order to do so, we have to choose a suitable vector $X_0 \in \mathfrak{D}_F$. Consider the following diagram:

$$\begin{array}{ccccc}
 & & \xrightarrow{\quad s \quad} & & \\
 (\mathbb{C}^F // N_\mathbb{C}^F) \setminus \{[0]\} & \xrightarrow{q_2} & \mathbb{C}_{\Delta_F}^F // N_\mathbb{C}^F \exp(i\mathfrak{s}) & \xrightarrow{q_1} & \mathbb{C}_{\Delta_F}^F // N_0^F \mathbb{C} \\
 \uparrow \chi_{\Sigma_F} & \textcircled{2} & \uparrow \chi'_{\Delta_F} & \textcircled{1} & \uparrow \chi_{\Delta_F} \\
 (\Psi_{\Sigma_F}^{-1}(0)/N^F) \setminus \{[0]\} & \xrightarrow{p_2} & (\Psi_{\Delta_F}^{-1}(0)/N^F) & \xrightarrow{p_1} & (\Psi_{\Delta_F}^{-1}(0)/N_0^F) \\
 & & \xleftarrow{\quad s' \quad} & &
 \end{array} \tag{37}$$

Consider first the diagram $\textcircled{1}$. The mapping χ_{Δ_F} is a diffeomorphism by the induction hypothesis. Proposition 2.7 can be applied in order to define the quotient $\mathbb{C}_{\Delta_F}^F // N_\mathbb{C}^F \exp(i\mathfrak{s})$: two points \underline{z} and \underline{w} in $\mathbb{C}_{\Delta_F}^F$ are equivalent if and only if

$$N^F \left(\overline{A_F \exp(i\mathfrak{s})\underline{z}} \right) \cap \overline{A_F \exp(i\mathfrak{s})\underline{w}} \neq \emptyset.$$

It was proved in [1] that the quotient $(\Psi_{\Delta_F}^{-1}(0)/N^F)$ is the link and in particular it is a space stratified by quasifolds of real dimension $2n-2p+1$. The inclusion $(\Psi_{\Delta_F}^{-1}(0) \hookrightarrow \mathbb{C}_{\Delta_F}^F)$ induces the continuous mapping χ'_{Δ_F} , which is bijective by Lemma 3.9, the inverse mapping being induced by the surjective mapping $\Xi_{\Delta_F} : \mathbb{C}_{\Delta_F}^F \rightarrow (\Psi_{\Delta_F}^{-1}(0))$. Lemma 3.12 implies that the mapping χ'_{Δ_F} is a diffeomorphism. It is easy to check that diagram $\textcircled{1}$ is commutative.

Consider now Diagram $\textcircled{2}$. The mapping q_2 is the natural projection, the mappings χ_{Σ_F} is defined by (34) and χ'_{Δ_F} is defined right above. We need to prove that χ_{Σ_F} is a diffeomorphism of stratified spaces. Let $\underline{x} \in \Psi_{\Sigma_F}^{-1}(0)$, $\underline{x} \neq 0$, we prove first that the mapping p_2 that makes the diagram commute is given by:

$$p_2([\underline{x}]) = [\underline{x}/|\underline{x}|_s]$$

where

$$|\underline{x}|_s = \sqrt{\sum_{j=1}^d s_j |x_j|^2}.$$

Let $\underline{z} \in \mathbb{C}^F$ such that

$$\{\underline{x}\} = \Psi_{\Sigma_F}^{-1}(0) \cap \overline{A\underline{z}}$$

therefore

$$\chi_{\Sigma_F}^{-1}([\underline{z}]) = [\underline{x}].$$

Notice that $\underline{z} \in \mathbb{C}_{\Delta_F}^F$, since the set of points in \mathbb{C}^F which are not equivalent to 0 under \sim_{N^F} are given exactly by $\mathbb{C}_{\Delta_F}^F$. Recall that the closure of the orbit $\overline{A \exp(i\mathfrak{s})\underline{z}}$, in $\mathbb{C}_{\Delta_F}^F$, is a union of orbits. From Lemma 3.9 the set $\Psi_{\Delta_F}(\overline{A \exp(i\mathfrak{s})\underline{z}})$ in $\mathfrak{n}_F^* \oplus \mathbb{R}$ is given by the union of the cones images of these orbits. The closure of $\Psi_{\Delta_F}(\overline{A \exp(i\mathfrak{s})\underline{z}})$ is a polyhedral cone in $\mathfrak{n}_F^* \oplus \mathbb{R}$, whose cone point is $(0, -1)$ by (32). Notice that $(0, -1) \notin \Psi_{\Delta_F}(\overline{A \exp(i\mathfrak{s})\underline{z}})$ but the half line $[0, -(1-t))$, $t > 0$, connecting the cone point with the origin, is contained in $\Psi_{\Delta_F}(\overline{A \exp(i\mathfrak{s})\underline{z}})$. Now observe that the points of the set $\overline{A \exp(i\mathfrak{s})\underline{z}}$ corresponding to the half line $[0, -(1-t))$ are exactly those given by $\overline{A \exp(i\mathfrak{s})\underline{z}} \cap \Psi_{\Sigma_F}^{-1}(0)$. This implies that $\chi_{\Sigma_F}^{-1}$ projects the orbit $A \exp(i\mathfrak{s})\underline{z}$ onto the curve in $\Psi_{\Sigma_F}^{-1}(0)/N^F$ given by $[\sqrt{t}x/|\underline{x}|_s]$, with $t > 0$. Therefore $p_2([\underline{x}]) = [\underline{x}/|\underline{x}|_s]$ and the quotient $X_{\Sigma_F} = \mathbb{C}^F//N_{\mathbb{C}}^F$ is a real cone over the quotient $\mathbb{C}_{\Delta_F}^F//N_{\mathbb{C}}^F \exp(i\mathfrak{s})$, which is the link of each point lying in the orbit \mathcal{T}_F . Since Σ_F , without the cone point, has the same depth of Δ_F , the mapping χ_{Σ_F} is a diffeomorphism away from the cone point by the induction hypothesis. The argument above shows that the mapping χ_{Σ_F} extends continuously to the cone point, thus giving a diffeomorphism of decomposed spaces (for a different proof see Remark 3.19).

We have thus proved all the three items above. Now we need to prove that

- *The complex and symplectic quotient are diffeomorphic.*

We work on the next diagram in order to check that the natural mapping

$$\chi_{\Delta} : \Psi^{-1}(0)/N \longrightarrow \mathbb{C}_{\Delta}^d//N_{\mathbb{C}},$$

for a polytope of depth n , is a diffeomorphism such that the symplectic structure of each stratum of $\Psi^{-1}(0)/N$ induces, via χ_{Δ} , a Kähler structure on the corresponding stratum of $\mathbb{C}_{\Delta}^d//N_{\mathbb{C}}$.

First of all it is easy to check that the mapping χ_{Δ} is continuous and bijective, the inverse mapping χ_{Δ}^{-1} is induced by Ξ_{Δ} . We need to prove that the inverse of χ_{Δ} is also continuous. This is true around regular points by the quoted result on simple polytopes [4, Thm.]. Consider the following diagram, which describes the local trivializations of the complex and symplectic quotients and their relationship:

$$\begin{array}{ccc} \frac{(\mathbb{C}^*)^{F^c} \times (\mathbb{C}^F//N_{\mathbb{C}}^F)}{N_{\mathbb{C}}/N_{\mathbb{C}}^F} & \xrightarrow{\quad} & ((\mathbb{C}^*)^{F^c} \times (\mathbb{C}^F))//N_{\mathbb{C}} \\ \uparrow & & \uparrow \chi_{\Delta} \\ \frac{\Psi^{-1}(0) \cap (\mathbb{C}^*)^{F^c} \times \Psi_{\Sigma_F}^{-1}(0)/N^F}{N/N_F} & \xrightarrow{h'} & (\Psi^{-1}(0) \cap (\mathbb{C}^F \times \mathbb{C}^*)^{F^c})/N \end{array} \quad (38)$$

We require the diagram to be commutative, this defines the mapping h' : more precisely, given $(\underline{x}, \underline{z}) \in (\Psi^{-1}(0) \cap (\mathbb{C}^*)^{F^c}) \times \Psi_{\Sigma_F}^{-1}(0)$, there exists a unique $a \in A$ such that

$a(\underline{x} + \underline{z}) \in \Psi^{-1}(0)$, the mapping h' takes $[\underline{x}, [\underline{z}]]$ to $[a(\underline{x} + \underline{z})]$. In order to prove that χ_Δ is closed it is enough to prove that h' is continuous.

Consider a closed subset C of the quotient $(\Psi^{-1}(0) \cap (\mathbb{C}^F \times \mathbb{C}^*)^{F^c})/N$ and let \tilde{C} be the N -invariant closed subset of $\Psi^{-1}(0) \cap (\mathbb{C}^F \times (\mathbb{C}^*)^{F^c})$ that projects onto C . We want to prove that the inverse image C_1 , via the mapping h' , of C , is closed in $(\Psi^{-1}(0) \cap (\mathbb{C}^*)^{F^c} \times \Psi_{\Sigma_F}^{-1}(0)/N^F)/(N/N_F)$. Let \tilde{C}_1 be the N -invariant subset of $(\Psi^{-1}(0) \cap (\mathbb{C}^*)^{F^c}) \times \Psi_{\Sigma_F}^{-1}(0)$ that projects onto C_1 . We prove that \tilde{C}_1 is closed. Let $(\underline{x}, \underline{z})$ be in the closure of \tilde{C}_1 , then there exists a sequence $(\underline{x}_n, \underline{z}_n) \in \tilde{C}_1$ converging to $(\underline{x}, \underline{z})$. Let G and G_n be the faces of Δ such that $\underline{x} + \underline{z} \in \mathbb{C}^{G^c}$ and $\underline{x}_n + \underline{z}_n \in \mathbb{C}^{G_n^c}$ and let \mathfrak{r} and \mathfrak{r}_n the orthogonal complement in \mathfrak{n} of $\mathfrak{n} \cap \mathbb{R}^G$ and $\mathfrak{n} \cap \mathbb{R}^{G_n}$ respectively. Then there exists a sequence $Y_n \in \mathfrak{r}_n$ such that $\exp(iY_n)(\underline{z}_n + \underline{x}_n) \in \tilde{C}$. The sequence $\exp(iY_n)(\underline{z}_n + \underline{x}_n)$ is in $\Psi^{-1}(0)$, therefore it admits a converging subsequence, that we denote again by $\exp(iY_n)(\underline{z}_n + \underline{x}_n)$. Let $\underline{w}_1 + \underline{w}_2 \in \mathbb{C}^F \times (\mathbb{C}^*)^{F^c}$ be its limit and let $Y \in \mathfrak{r}$ such that $\exp(iY)(\underline{x} + \underline{z}) \in \Psi^{-1}(0)$. Notice first that for $j \notin I_F$ the sequence $\exp(\langle Y_n, \iota^*(e_j^*) \rangle)$ converges to $\exp(\langle Y, \iota^*(e_j^*) \rangle)$ and $(\underline{w}_1 + \underline{w}_2)_j = \exp(\langle Y, \iota^*(e_j^*) \rangle)(\underline{x} + \underline{z})_j$. Observe now that there exists a face H such that $(\underline{w}_1 + \underline{w}_2) \in (\mathbb{C}^*)^{H^c}$. We have

$$I_H \subset I_G \subset I_F.$$

Suppose that I_H is a proper subset of I_G . Then take $\xi \in H \setminus G$ and $\eta \in G$: the coefficients

$$c_j = \langle \xi, X_j \rangle - \langle \eta, X_j \rangle$$

have the following properties:

- a) $c_j = \lambda_j - \lambda_j = 0$, for $j \in I_H$
- b) $c_j = \langle \xi, X_j \rangle - \lambda_j > 0$ for $j \in I_G \setminus I_H$.

We define on the subset $\mathbb{C}^H \times (\mathbb{C}^*)^{H^c}$ the continuous function

$$P(z) = \prod_{j=1}^d |z_j|^{c_j}$$

We have that

$$\lim_n P(\exp(iY_n)(\underline{x}_n + \underline{z}_n)) = P(\underline{w}_1 + \underline{w}_2) \neq 0$$

on the other hand the function P is invariant under the action on $N_{\mathbb{C}}$, therefore

$$\lim_n P(\exp(iY_n)(\underline{x}_n + \underline{z}_n)) = \lim_n P(\underline{x}_n + \underline{z}_n) = P(\underline{x} + \underline{z}) = 0.$$

It follows that $H = G$ and $\underline{w}_1 + \underline{w}_2 = \exp(iY)(\underline{x} + \underline{z})$. This proves that $\underline{x} + \underline{z} \in \tilde{C}_1$ is closed and therefore h' is continuous. \square

Corollary 3.17 *The space X_Δ is compact.*

Remark 3.18 Theorem 3.16 improves the result of local triviality that was found in [1]: diagram (38) implies that the link *does not* depend on the point but just on the stratum, moreover the local trivialization is of the form

$$(\tilde{B}_F \times C(L_F))/\tilde{\Gamma}.$$

where $\tilde{B}_F/\tilde{\Gamma}$ is a chart that covers the whole stratum \mathcal{T}_F .

Remark 3.19 We can date back to the work [14] the study of the relationship between symplectic and complex quotients. We have given a direct and self contained proof of the fact the two quotients are homeomorphic, to us one of the hardest points in the proof is to show that the mapping χ_{Δ}^{-1} is continuous. This has been proved by F. Kirwan in [15], in the case of a reductive group acting on a compact Kähler manifold with finite isotropy and in broad generality by F. Loose and P. Heinzner in [13]. In the case of a torus acting on a vector space, with homogeneous moment mapping, the result was proved by A. Neeman by showing that the zero set of the moment mapping is a deformation retraction of the vector space [17]. Based on Neeman's result, G. Schwarz, in his review [20], gave a proof for the general case of reductive groups. We point out that Neeman's result apply directly to our context in the case of the cone X_{Σ_F} , since the proof is based on estimates and computations at the Lie algebra level.

Remark 3.20 Observe that the quotient X_{Δ} , together with its complex structure, only depends on our choice of generators of the fan and on the choice of quasilattice, whilst the symplectic structure depends of course on the polytope, as in the rational case.

References

- [1] F. Battaglia, Convex Polytopes and Quasilattices from the Symplectic Viewpoint, *Commun. Math. Phys.* 269, 283-310 (2007).
- [2] F. Battaglia, Complex quotients by nonclosed groups and their stratifications, *C. R. Math. Rep. Acad. Sci. Canada* Vol. 29 2007, pp. 33-40.
- [3] F. Battaglia, Betti numbers of the geometric spaces associated to non-rational simple convex polytopes, preprint arXiv:1004.4763v1 [math.AG] (2010).
- [4] F. Battaglia, E. Prato, Generalized toric varieties for simple nonrational convex polytopes, *Intern. Math. Res. Notices* 24 (2001), 1315-1337.
- [5] F. Battaglia, E. Prato, The Symplectic Geometry of Penrose Rhombus Tilings, *J. Symplectic Geom.* 6 (2008), 139-158.
- [6] F. Battaglia, E. Prato, The Symplectic Penrose Kite, preprint arXiv:0711.1642v3 [math.SG] (2007), to appear in *Commun. Math. Phys.*
- [7] F. Battaglia, E. Prato, Ammann Tilings in Symplectic Geometry, preprint arXiv:1004.2471v2 [math.SG] (2010).
- [8] D. Cox, The homogeneous coordinate ring of a toric variety, *J. Algebraic Geom.* 4 (1995), no. 1, 17-50.
- [9] T. Delzant, Hamiltoniens périodiques et image convexe de l'application moment, *Bull. Soc. Math. France* 116 (1988), 315-339.

- [10] M. Goresky, R. MacPherson, *Stratified Morse Theory*, Springer Verlag, New York, 1988.
- [11] B. Grünbaum, *Convex polytopes*, Graduate Text in Math. 221, Springer, 2003.
- [12] V. Guillemin, *Moment maps and combinatorial invariants of Hamiltonian T^n -spaces*, Progress in Mathematics 122, Birkhäuser, Boston, 1994.
- [13] Heinzner, P.; Loose, F. Reduction of complex Hamiltonian G -spaces. *Geom. Funct. Anal.* 4 (1994), no. 3, 288–297.
- [14] G. Kempf and L. Ness, The length of vectors in representation spaces, *Algebraic Geometry, Proceedings, Copenhagen, 1978*, Lecture Notes in Math. 732 (1979), 233–244.
- [15] F. Kirwan, *Cohomology of quotients in symplectic and algebraic geometry*, Mathematical Notes 31, Princeton University Press, 1984.
- [16] I. Musson, Differential operators on toric varieties. *J. Pure Appl. Algebra* 95 (1994), no. 3, 303–315.
- [17] A. Neeman The topology of quotient varieties. *Ann. of Math. (2)* 122 (1985), no. 3, 419–459.
- [18] E. Prato, Simple non-rational convex polytopes via symplectic geometry, *Topology* 40 (2001), 961–975.
- [19] J. Richter-Gebert, *Realization Spaces of Polytopes*, Lecture Notes in Mathematics 1643, Springer, 1996.
- [20] G. Schwarz, The topology of algebraic quotients, in *Topological Methods in Algebraic Transformation Groups*, (eds. H. Kraft, T. Petrie, G. W. Schwarz), Progress in Mathematics vol. 10, Birkhäuser Verlag, Basel-Boston, 1989, pp. 135–152.
- [21] M. Senechal, *Quasicrystals and Geometry*, Cambridge University Press, Cambridge, 1995.
- [22] W. Steurer, S. Deloudi, *Cristallography for quasicrystals*, Springer Series in Materials Science, vol. 29 (2009).
- [23] G. Ziegler, Non-rational configurations, polytopes, and surfaces, *Math. Intelligencer* 30, Number 3 (2008), 36–42.